

Beliefs Aggregation and Return Predictability

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March 5, 2021

Abstract

We study return predictability using a model of speculative trading among competitive traders who agree to disagree about the precision of their private information. Although traders apply Bayes Law consistently, returns are predictable. In addition to trading on long-term fundamental value, traders also trade on perceived short-term opportunities arising from foreseen future disagreement, as in a Keynesian beauty contest. Contradicting conventional wisdom, this short-term speculation dampens price fluctuations and leads to time-series momentum. Model calibration generates quantitatively realistic patterns of return dynamics. Consistent with empirical evidence, our model predicts more pronounced momentum for stocks with higher trading volume.

JEL: B41, D8, G02, G12, G14.

Keywords: return predictability, market microstructure, market efficiency, momentum, mean-reversion, anomalies, agreement to disagree.

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Although researchers have documented short-run time-series momentum in equity returns, they have found it notoriously difficult to explain theoretically why this pattern can occur in a competitive market. We present a dynamic model of competitive trading based on flows of new private information. The model generates time-series momentum endogenously by relaxing the rationality assumption in only a minimal way.

As new information arrives, all traders apply Bayes Law consistently and optimize correctly, with the single exception that each trader symmetrically assigns a higher precision to his own signal than to other traders' signals. As a result of this relative overconfidence, each trader anticipates that, upon observing new information in the future, others will correct their mistakes by adjusting their valuations towards unconditional levels. To profit from this perceived short-run return predictability, traders make speculative trades which bet against other traders' expectations. This dampens prices relative to traders' own long-term valuations and leads to time series momentum even if traders' signals are empirically correct on average.

For example, suppose all traders have positive signals such that they all agree that the fundamental value is \$30. Since each trader believes that others are putting too much weight on both their current and future information, each trader believes that other traders will revise their expectations downward in the future. As a result, each trader would want to trade on this short-term profit opportunity by selling today at price \$30 and plan to buy at better (lower) prices later. This internalization of future disagreement reduces the price today below \$30, even when all traders correctly agree that \$30 is an accurate estimate of fundamental value. As Keynes puts it, "it is not sensible to pay 25 for an investment of which you believe the prospective yield to justify value of 30, if you also believe that the market will value it at 20 three months hence."

While this short-term trading faithfully reflects the logic of a Keynesian beauty contest, its empirical implication contradicts Keynes's unmodeled intuition that short-term speculation amplifies volatility and leads to mean reversion in prices. In contrast, our theory implies dampened price fluctuations, dampened volatility, and therefore momentum—exactly the opposite of what Keynes expected. We first confirm our insight using a two-period model with closed-form solution. We then obtain a quasi-closed-form numerical solution to an infinite-horizon discrete-time model. Price dampening is more pronounced in the infinite-horizon dynamic trading model than in the two-period model. The intuition that prices are dampened because of speculative trading suggests the hypothesis that increasing the frequency of trading will increase momentum. Indeed, we show that increased trading frequency magnifies speculative

trading in a Keynesian beauty contest and leads to larger momentum.

With belief heterogeneity, empirical prices and quantities depend on both traders' beliefs and the empirically correct model specification which ultimately govern the dynamics of fundamentals and information. For example, when traders' beliefs about the decay rates of signals are different from the empirically correct parameter, the expected returns are described by a structural model which is linear in state variables including the current levels of prices and dividends as well as exponentially weighted averages of these variable in the past. We calibrate model parameters and obtain positive return autocorrelation over short periods of one to two years and negative autocorrelation over longer horizons, matching empirically observed return patterns. Time-series return momentum arises from price dampening, and return mean reversion arises when traders believe that the signals decay at a slower rate than the empirically correct parameter implies.

Our results contribute to the theoretical literature concerning whether differences in beliefs or expectations can generate price drift (defined as positive return autocorrelation). The rational expectations paradigm of Muth (1961) implies that price changes are unpredictable because prices aggregate fundamental information correctly when traders are correct on average, even when individual traders make mistakes. It may thus seem reasonable to attribute return predictability, including momentum, to irrational behavior or bounded rationality. Yet, when return anomalies are motivated by behavioral biases, Fama (1998) suggests that a Pandora's box is opened, undermining modeling parsimony by enabling one plethora of behavioral biases to explain another plethora of anomalies. We therefore explore the suggestion of Morris (1995) for "dropping the common prior assumption from otherwise rational behavior," based on the idea that even rational agents may have heterogeneous beliefs. We do this by assuming overconfidence, like Aumann (1976).

Allen, Morris, and Shin (2006) attribute price drift to iterating expectations in a market where traders are assumed to share a common prior but observe different information. Yet, Banerjee, Kaniel, and Kremer (2009) point out that in such a noisy rational expectations model, when traders are allowed to learn from price, returns should exhibit mean reversion, not momentum, due to the noise in prices. Banerjee, Kaniel, and Kremer (2009) also show that, for heterogeneous beliefs to generate price drift, it is necessary for traders to disagree not only about the joint distribution of signals and fundamental value but also about the joint distribution of signals themselves. They present a three-period model in which traders receive signals only at

date 0, and aggregate noisy asset supply follows a random walk. Yet, Kyle and Wang (2020) show that, contrary to claims made in their paper, there is no positive return autocorrelation under these assumptions.

Our two-period model and infinite-horizon model satisfy both of the necessary conditions for price drift pointed out by Banerjee, Kaniel, and Kremer (2009). Our model generates positive return autocorrelation by assuming that traders receive new signals over time, disagree about how to interpret the signals both in the present and in the future, and understand correctly how this disagreement plays out in prices, reflecting trading in the real world. In our framework, these provide sufficient conditions to generate price dampening and return momentum. The return predictability is related to the perceived order flow predictability due to traders' short-term speculative trading based on disagreement about the information content of future signals.

Our model is similar to Kyle, Obizhaeva, and Wang (2018), who show how strategic traders rationally smooth out their trading in continuous time to reduce permanent and temporary price impact. It is different in that we assume a competitive model of trading in discrete time. Assuming perfect competition and discrete trading allows us to make direct comparisons with the existing literature, which also assumes perfect competition and discrete time. Assuming discrete-time trading also allows us to show that momentum is more pronounced when trading opportunities are more frequent.

Hong and Stein (1999), Barberis and Shleifer (2003), and Greenwood and Shleifer (2014) generate return predictability by assuming that traders follow simple trading rules and do not extract information from prices. Gruber (1996), Lou (2012), and Vayanos and Woolley (2013) derive time-series momentum from the aggregate amount of money chasing returns. Detemple and Murthy (1994), Basak (2005), Jouini and Napp (2007), Dumas, Kurshev, and Uppal (2009), Xiong and Yan (2010), Cujean and Hasler (2014), Atmaz and Basak (2015), and Ottaviani and Sorensen (2015) derive return predictability from the interaction of beliefs aggregation and wealth effects. Andrei and Cujean (2017) study word-of-mouth communication instead of beliefs aggregation as a mechanism that generates return predictability. In our model, return predictability is unrelated to traders following simple rules, wealth effects, the flow of money into the market, and word of mouth communication. Instead, we assume traders follow optimal strategies, there are no wealth effects, the asset is in zero net supply, and traders infer other traders' information from prices.

Consistent with empirical evidence, our model implies time-series momentum to be more significant when disagreement among traders is larger and assets are more actively traded. Lee and Swaminathan (2000) and Cremers and Pareek (2014) find momentum to be stronger for stocks with higher volume and more short-term trading. Zhang (2006) and Verardo (2009) show that momentum returns are larger for stocks with higher analysts' disagreement. Moskowitz, Ooi, and Pedersen (2012) document time-series momentum in liquid futures contracts. Our findings may also be relevant for empirical research on return predictability in models with heterogenous beliefs, such as Greenwood and Shleifer (2014) and Buraschi, Piatti, and Whelan (2016).

This paper is structured as follows. Section 1 presents a two-period model and describes intuition behind price dampening. Section 2 considers a competitive dynamic model with discrete trading and further clarifies mechanisms generating price dampening. Section 3 analyzes predictions for holding-period returns. Section 4 calibrates the model parameters. Section 5 concludes. Proofs are in the Appendix.

1 A Two-Period Model

To explain the economic intuition behind price dampening and time-series momentum, we first present a simple two-period model.

1.1 Model Setup

Consider a two-period model of competitive trading among N traders with different beliefs. A risky asset with random liquidation value $v \sim \mathcal{N}(0, 1/\tau_v)$ is traded for a safe numeraire asset. The total supply of the risky asset is zero.

Traders trade in periods 1 and 2, and the payoff v is realized at period 3. At periods $t = 1$ and $t = 2$, each trader n observes private signals i_{nt} ,

$$i_{nt} := \tau_n^{1/2} (\tau_v^{1/2} v) + e_{nt},$$

with $e_{nt} \sim \mathcal{N}(0, 1)$. The asset payoff v and $2N$ private signal errors $e_{1t}, \dots, e_{Nt}$ with $t = 1, 2$ are independently distributed.¹

¹We deviate from the standard approach in the literature, which assumes that a signal has the form value-

In equilibrium, prices are functions of the average of traders' private signals.² Therefore, each trader n can infer from the equilibrium price the average of other traders' private signals, denoted i_{-nt} :

$$i_{-nt} := \frac{1}{N-1} \sum_{m=1, m \neq n}^N i_{mt}.$$

To generate trading, we assume that traders agree to disagree about the precisions of private signals τ_n . Each trader is “relatively overconfident,” believing his own signal has a high precision $\tau_n = \tau_H$ and other traders' signals have low precision $\tau_m = \tau_L$ for $m \neq n$, with $\tau_H > \tau_L \geq 0$.

Each trader has the same exponential utility function with constant-absolute-risk-aversion parameter A . At each period $t = 1, 2$, trader n 's problem is to choose the optimal inventory S_{nt} to maximize his expected exponential utility function of terminal wealth at time $t = 3$:

$$E_t^n [-e^{-A W_{n3}}], \quad (1)$$

where $E_t^n [\dots]$ denotes trader n 's expectation conditional on all signals at period $t = 1, 2$, and the terminal wealth W_{n3} is

$$W_{n3} = W_{n1} + (P_2 - P_1) S_{n1} + (v - P_2) S_{n2}.$$

The prices P_1 and P_2 are set to clear markets each period.

Let $\text{Var}_t^n [\dots]$ denote trader n 's variance operator conditional on all signals at period $t = 1, 2$. The projection theorem for normally distributed random variables implies that the conditional error variance is related to signal precision τ_t at time $t = 1$ and 2 by

$$\tau_t := (\text{Var}_t^n [v])^{-1} = \tau_v (1 + (\tau_H + (N-1) \tau_L) t). \quad (2)$$

plus-noise, and its informativeness is given by the variance of the noise term. Instead, we rescale signals by the standard deviation of noise terms $\tau_n^{1/2}$, so that the coefficient on the diffusion term is exactly one. This makes it harder for a trader to learn about the precision of a signal from its sample variance. If we assume $i_{nt} := (\tau_n + 1)^{-1/2} (\tau_n^{1/2} (\tau_v^{1/2} v) + e_{nt})$, with trader n believing that his signal's precision is $\tau_n = \tau_H$ and other traders' signal precision is $\tau_m = \tau_L$, $m \neq n$, then it can be shown that $\text{Var}(i_{nt}) = \text{Var}(i_{mt}) = 1$, and traders do not disagree about the variance of signals. We show that the dynamic dampening effect arises and return still exhibits momentum in this setting because traders disagree about the coefficient of v in the signals, $\tau_H^{1/2} (\tau_H + 1)^{-1/2} > \tau_L^{1/2} (\tau_L + 1)^{-1/2}$. To make the two-period model comparable to our continuous-time setting where a trader can estimate the diffusion variance (quadratic variation) with high accuracy (see footnote 6), we rescale signals so that the coefficient on the noise term is exactly one in the two-period setting.

²For expositional simplicity, we assume there are no public signals in the two-period model. Including public signals does not change any of our main results.

Trader n 's expected value of v at time $t = 1, 2$ is

$$E_t^n[v] = \frac{\tau_v^{1/2}}{\tau_t} \left(\tau_H^{1/2} \sum_{s=1}^t i_{ns} + (N-1) \tau_L^{1/2} \sum_{s=1}^t i_{-ns} \right),$$

which puts weight proportional to $\tau_H^{1/2}$ on trader n 's own private signals i_{nt} and to $\tau_L^{1/2}$ on others' private signals i_{mt} , $m \neq n$.

1.2 The Equilibrium with Myopic Conditional Mean-Variance Optimizers

We first solve for equilibrium prices when traders are myopic and hold conditional mean-variance optimal portfolios.³

Theorem 1 (Equilibrium with Myopic Conditional Mean-Variance Optimizers). *At $t = 1, 2$, (1) trader n 's myopic mean-variance optimal inventory is*

$$\check{S}_{n1} = \frac{E_1^n[\check{P}_2] - \check{P}_1}{A \text{Var}_1^n[\check{P}_2]}, \quad \check{S}_{n2} = \frac{E_2^n[v] - \check{P}_2}{A \text{Var}_2^n[v]}, \quad (3)$$

and (2) the prices $\check{P}_1$ and $\check{P}_2$ are

$$\check{P}_1 = \frac{1}{N} \sum_{n=1}^N E_1^n[\check{P}_2] = \check{C}_g \frac{1}{N} \sum_{n=1}^N E_1^n[v], \quad \check{P}_2 = \frac{1}{N} \sum_{n=1}^N E_2^n[v], \quad (4)$$

where the coefficient $\check{C}_g$ satisfies $0 < \check{C}_g < 1$ and is given by

$$\check{C}_g := 1 - \frac{\tau_v}{\tau_2} \left(1 - \frac{1}{N}\right) \left(\tau_H^{1/2} - \tau_L^{1/2}\right)^2. \quad (5)$$

Proof: The proof is in Appendix A.1. □

In the case with myopic mean-variance optimizers, traders do not have any hedging demand. Equation (3) in Theorem 1 implies that each trader's optimal inventory at $t = 1$ is his short-term speculative position. Equation (4) in Theorem 1 implies that the equilibrium price

³The results are equivalent to the results of assuming that traders must consume their period t trading gains at period t ($t = 2, 3$), so they choose optimal inventory S_{nt} ($t = 1, 2$) to maximize $E_1^n[-e^{-A W_{n2}}]$ and $E_2^n[-e^{-A W_{n3}}]$, where W_{n2} and W_{n3} are $W_{n2} = W_{n1} + (P_2 - P_1) S_{n1}$ and $W_{n3} = (v - P_2) S_{n2}$.

at $t = 1$ is a weighted average of traders' expectations of the fundamental value ν with weights summing to a constant $\check{C}_g$, which is less than one.

In our setting, traders agree to disagree about the interpretation of signals in the present, and they also agree to disagree about how to interpret the signals arriving in the future. More specifically, each trader n observes both i_{nt} and the average of others' signal i_{-nt} ($t = 1, 2$) and he believes that both his current and future signals have a high precision and other traders' current and future signals have low precision. As a result, even if all traders happen to agree about the fundamental value at time $t = 1$ (so $E_1^n[\nu] = E_1^m[\nu] > 0$ for all $m \neq n$), trader n 's current expectation of the average of others' future expectations $E_1^n[\frac{1}{N-1} \sum_{m=1, m \neq n}^N E_2^m[\nu]]$ is lower than his own current and future expectations $E_t^n[\nu]$, $t = 1, 2$. To see this, it can be shown that, trader n 's expectations of others' expectations next period can be written as

$$E_1^n\left[\frac{1}{N-1} \sum_{m=1, m \neq n}^N E_2^m[\nu]\right] = \left(1 - \left(\tau_H^{1/2} - \tau_L^{1/2}\right)^2 \frac{\tau_\nu}{\tau_2}\right) E_1^n[\nu] - \frac{\tau_1}{\tau_2} \left(E_1^n[\nu] - \frac{1}{N-1} \sum_{m=1, m \neq n}^N E_1^m[\nu]\right). \quad (6)$$

At time $t = 1$, trader n 's estimate of $\check{P}_2$ is

$$E_1^n[\check{P}_2] = E_1^n\left[\frac{1}{N} \sum_{n=1}^N E_2^n[\nu]\right] = \frac{\tau_1}{\tau_2} \frac{1}{N} \sum_{n=1}^N E_1^n[\nu] + \frac{\tau_\nu}{N\tau_2} \left(\tau_H^{1/2} + (N-1)\tau_L^{1/2}\right)^2 E_1^n[\nu].$$

If all traders happen to agree about the fundamental value at time $t = 1$ (i.e., $E_1^n[\nu] = E_1^m[\nu] > 0$, for all $m \neq n$), we obtain

$$E_1^n[\check{P}_2] = \left(1 - \frac{\tau_\nu}{\tau_2} \left(1 - \frac{1}{N}\right) \left(\tau_H^{1/2} - \tau_L^{1/2}\right)^2\right) \frac{1}{N} \sum_{n=1}^N E_1^n[\nu] < \frac{1}{N} \sum_{n=1}^N E_1^n[\nu]. \quad (7)$$

Equation (7) implies that each trader n expects the price to fall below fundamental value in the next period. In addition, at $t = 1$, each trader n expects his own optimal inventory and the average of others' optimal inventory next period to be

$$\begin{aligned} E_1^n[\check{S}_{n2}] &= \frac{E_1^n[E_2^n[\nu] - \check{P}_2]}{A \text{Var}_2^n[\nu]} \\ &= \frac{(N-1)\tau_\nu}{A N} \left(\tau_H^{1/2} - \tau_L^{1/2}\right)^2 E_1^n[\nu] + \frac{(N-1)\tau_1}{A N} \left(E_1^n[\nu] - \frac{1}{N-1} \sum_{m=1, m \neq n}^N E_1^m[\nu]\right), \end{aligned} \quad (8)$$

$$E_1^n \left[\frac{1}{N-1} \sum_{m=1, m \neq n}^N \check{S}_{m2} \right] = \frac{E_1^n \left[\frac{1}{N-1} \sum_{m=1, m \neq n}^N E_2^m [\nu] - \check{P}_2 \right]}{A \text{Var}_2^n [\nu]} = -\frac{1}{N-1} E_1^n [\check{S}_{n2}]. \quad (9)$$

Equation (8) implies that, if all traders happen to agree about the fundamental value at time $t = 1$ (so $E_1^n [\nu] = E_1^m [\nu] > 0$, for all $m \neq n$), then each trader expects himself to buy and others to sell next period because each trader expects prices to fall in the next period. Each trader believes that others make the mistake of attributing too much precision to their current signals. Each trader anticipates that, upon observing new information in the future, others will adjust their valuations towards unconditional levels, and thus each trader expects others to sell and prices to fall at $t = 2$.

Therefore, even if the price at $t = 1$ was equal to the consensus fundamental valuation ($\check{P}_1 = \frac{1}{N} \sum_{n=1}^N E_1^n [\nu]$), all traders would want to hold short positions ($\check{S}_{n1} < 0$) at $t = 1$ because all of them would expect prices to fall below fundamental value in the next period. In the case with myopic mean-variance optimizers, the equilibrium price $\check{P}_1$ is traders' average expectations of next period price $\check{P}_2$. Since all traders expect prices to fall below fundamental value in the next period (equation (7)), it follows that

$$\begin{aligned} \check{P}_1 &= \frac{1}{N} \sum_{n=1}^N E_1^n [\check{P}_2] = \left(1 - \frac{\tau_\nu}{\tau_2} \left(1 - \frac{1}{N} \right) (\tau_H^{1/2} - \tau_L^{1/2})^2 \right) \frac{1}{N} \sum_{n=1}^N E_1^n [\nu] \\ &= \check{C}_g \frac{1}{N} \sum_{n=1}^N E_1^n [\nu], \end{aligned} \quad (10)$$

where the dampening factor $\check{C}_g$ defined in equation (5) satisfies $0 < \check{C}_g < 1$.

To summarize, all traders internalize at the present their perceived disagreement with one another about future valuations. If prices were equal to the consensus fundamental valuation, all traders would want to hold short positions at $t = 1$. As a result, the equilibrium price becomes dampened in the first period relative to traders' average contemporaneous expectations about the fundamental value, $\frac{1}{N} \sum_{n=1}^N E_1^n [\nu]$. The price dampening effect increases ($\check{C}_g$ is lower) in the magnitude of disagreement τ_H/τ_L . Since traders do not have any hedging demand in the setting with myopic conditional mean-variance optimizers, it implies that the key driving force of the price dampening is traders' short-term speculation due to their disagreement about future valuations.

1.3 The Equilibrium of the Two-Period Model

We next solve for the general two-period trading model where trader n chooses the inventory S_{nt} ($t = 1, 2$) to solve his optimization problem (1). The following theorem characterizes the equilibrium in the two-period model.

Theorem 2 (Equilibrium of the Two-Period Model). (1) At $t = 2$, trader n 's optimal inventory S_{n2} and the price P_2 are given as

$$S_{n2} = \frac{E_2^n[v] - P_2}{A \text{Var}_2^n[v]}, \quad P_2 = \frac{1}{N} \sum_{n=1}^N E_2^n[v].$$

(2) At $t = 1$, trader n 's optimal inventory is

$$S_{n1} = \frac{\text{Cov}_1^n[E_2^n[v], E_2^n[v] - P_2] + \text{Var}_2^n[v]}{A (\text{Var}_1^n[P_2] (\text{Var}_1^n[E_2^n[v]] + \text{Var}_2^n[v]) - (\text{Cov}_1^n[P_2, E_2^n[v]])^2)} \times \left((E_1^n[P_2] - P_1) - \frac{\text{Cov}_1^n[P_2, E_2^n[v] - P_2]}{\text{Cov}_1^n[E_2^n[v], E_2^n[v] - P_2] + \text{Var}_2^n[v]} (E_1^n[v] - P_1) \right). \quad (11)$$

The price P_1 is a weighted average of traders' expectations of the fundamental value v with weights summing to a constant C_g , which is less than one:

$$\begin{aligned} P_1 &= \frac{1}{N} \sum_{n=1}^N E_1^n[P_2] - \frac{\text{Cov}_1^n[P_2, E_2^n[v] - P_2]}{\text{Var}_1^n[E_2^n[v] - P_2] + \text{Var}_2^n[v]} \left(\frac{1}{N} \sum_{n=1}^N E_1^n[v] - \frac{1}{N} \sum_{n=1}^N E_1^n[P_2] \right) \\ &= C_g \frac{1}{N} \sum_{n=1}^N E_1^n[v]. \end{aligned} \quad (12)$$

Proof: The proof is in Appendix A.2. Equation (A-4) is a closed-form expression for the dampening factor C_g . $\square$

Trader n 's optimal inventory at $t = 1$ (equation (11)) can be decomposed into two components: (i) A long-term position (buy and hold position) proportional to $E_1^n[v] - P_1$, similar to the holdings in a one-period model; and (2) A short term speculative position proportional to $E_1^n[P_2] - P_1$.

Equation (11) in Theorem 2 implies that trader n 's long-term position is proportional to

$E_1^n[\nu] - P_1$ and the proportionality constant is negative since

$$\begin{aligned} \text{Var}_1^n[P_2] \text{Var}_1^n[E_2^n[\nu]] &> (\text{Cov}_1^n[P_2, E_2^n[\nu]])^2, \\ \text{Cov}_1^n[P_2, E_2^n[\nu] - P_2] &= \frac{\tau_v^2(N-1)}{\tau_1\tau_2^2N^2} \left(\tau_H^{1/2} + (N-1)\tau_L^{1/2} \right)^2 (\tau_H^{1/2} - \tau_L^{1/2})^2 > 0, \end{aligned} \quad (13)$$

with parameters τ_1 and τ_2 defined as in equation (2).

As explained in the previous section, traders agree to disagree about how to interpret the information today, and they understand that they may continue to disagree with one another about how to interpret new information in the future. Even if the price at $t = 1$ were equal to the consensus fundamental valuation, all traders would expect prices to fall below fundamental value in the next period. As a result, even if trader n 's estimate of the fundamental value is greater than the price at $t = 1$ ($E_1^n[\nu] > P_1$), traders' optimal "long-term position" is negative.

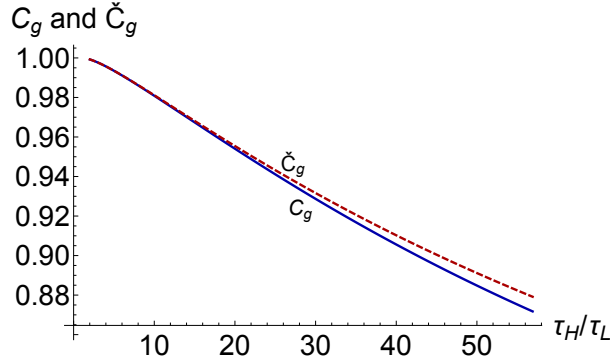
Equation (11) in Theorem 2 also implies that trader n 's short-term position is proportional to $E_1^n[P_2] - P_1$ and the proportionality constant is positive. Since $P_2 = \check{P}_2 = \frac{1}{N} \sum_{n=1}^N E_1^n[\nu]$, similar to the case with myopic mean-variance optimizers (equations (7) and (8)), if all traders happen to agree about the fundamental value at time $t = 1$ (so $E_1^n[\nu] = E_1^m[\nu] > 0$, for all $m \neq n$), then

$$E_1^n[P_2] = E_1^n[\check{P}_2] = \left(1 - \frac{\tau_v}{\tau_2} \left(1 - \frac{1}{N} \right) (\tau_H^{1/2} - \tau_L^{1/2})^2 \right) \frac{1}{N} \sum_{n=1}^N E_1^n[\nu] < \frac{1}{N} \sum_{n=1}^N E_1^n[\nu], \quad (14)$$

and each trader n expects himself to buy and others to sell next period because each trader expects prices to fall in the next period.

Therefore, if the price at $t = 1$ were equal to the consensus fundamental valuation ($P_1 = \frac{1}{N} \sum_{n=1}^N E_1^n[\nu]$), all traders would want to hold short positions due to short-term speculation ($E_1^n[P_2] - P_1 < 0$). It follows that both the long-term position and the short-term speculative position are negative at $t = 1$.

Equation (12) in Theorem 2 shows that the price P_1 is a weighted average of traders' estimates of the fundamental value of the asset ($\frac{1}{N} \sum_{n=1}^N E_1^n[\nu]$) and traders' estimates of next period's price ($\frac{1}{N} \sum_{n=1}^N E_1^n[P_2]$). Because traders expect prices to fall below fundamental value in the next period, the weight on $\frac{1}{N} \sum_{n=1}^N E_1^n[\nu]$ is negative while the weight on $\frac{1}{N} \sum_{n=1}^N E_1^n[P_2]$ is

Figure 1: C_g and $\check{C}_g$ against τ_H/τ_L .

greater than one. We have

$$\begin{aligned}
 P_1 &= \frac{1}{N} \sum_{n=1}^N E_1^n [P_2] - \frac{\text{Cov}_1^n [P_2, E_2^n [\nu] - P_2]}{\text{Var}_1^n [E_2^n [\nu] - P_2] + \text{Var}_2^n [\nu]} \left(\frac{1}{N} \sum_{n=1}^N E_1^n [\nu] - \frac{1}{N} \sum_{n=1}^N E_1^n [P_2] \right) \\
 &= \check{C}_g \frac{1}{N} \sum_{n=1}^N E_1^n [\nu] - \frac{\tau_\nu}{\tau_2} \left(1 - \frac{1}{N} \right) \left(\tau_H^{1/2} - \tau_L^{1/2} \right)^2 \frac{\text{Cov}_1^n [P_2, E_2^n [\nu] - P_2]}{\text{Var}_1^n [E_2^n [\nu] - P_2] + \text{Var}_2^n [\nu]} \frac{1}{N} \sum_{n=1}^N E_1^n [\nu] \quad (15) \\
 &= C_g \frac{1}{N} \sum_{n=1}^N E_1^n [\nu].
 \end{aligned}$$

Different from the case with myopic mean-variance optimizers (equation (10)), the first line of equation (15) states that price P_1 is the average expectation of the price P_2 , adjusted for the extra trading due to hedging demand related to traders' expectation at $t = 1$ of the next period's return ($E_1^n [E_2^n [\nu] - P_2]$). The first term in the second line is the same as the second line of equation (10) in the myopic case. It shows that the average expectation of P_2 at $t = 1$ is dampened relative to the average expectation of fundamentals, $\frac{1}{N} \sum_{n=1}^N E_1^n [\nu]$. The last line combines terms and shows that price P_1 is further dampened relative to the average expectations of fundamentals, $\frac{1}{N} \sum_{n=1}^N E_1^n [\nu]$, compared to the myopic case; we prove analytically that the coefficient C_g satisfies $0 < C_g < 1$. We have the following Proposition.

Proposition 1. *The price dampening factor C_g satisfies $0 < C_g < \check{C}_g < 1$. In addition, both C_g and $\check{C}_g$ decrease in the disagreement level τ_H/τ_L , holding fixed the total precision $\tau_H + (N - 1)\tau_L$.*

Proof: The proof is in Appendix A.3. □

Figure 1 shows that price dampening factors C_g and $\check{C}_g$ decrease in the disagreement level

τ_H/τ_L .⁴ This is because traders trade more aggressively against one another with greater disagreement. As illustrated in Figure 1, the difference between C_g and $\check{C}_g$ is quite small for a large range of parameter values. This suggests that the magnitude of the additional dampening effect from hedging demand at $t = 1$ is small, and the main driving force of the dampening effect comes from traders' short-term speculation on their disagreement about future valuation.

As in the simple myopic case, traders agree to disagree about the interpretation of signals in the present, and they also agree to disagree about how to interpret the signals arriving in the future. Each trader expects prices to fall below fundamental value in the next period. Even if all traders happen to agree about the fundamental value at time $t = 1$ (assume $E_1^n[\nu] = E_1^m[\nu] > 0$, for all $m \neq n$), trader n 's current expectation of the average of others' expectations $E_1^n[\frac{1}{N-1} \sum_{m=1, m \neq n}^N E_2^m[\nu]]$ is lower than his own expectation of fundamental $E_1^n[\nu]$. All traders internalize at the present their perceived disagreement with one another about the future valuations. If prices were equal to the consensus fundamental valuation, all traders would want to hold short positions because all of them would expect prices to fall below fundamental value in the next period, as other traders are expected to revise their expectations. In aggregate, the equilibrium price becomes dampened in the first period relative to traders' average contemporaneous expectations about the fundamental value, $\frac{1}{N} \sum_{n=1}^N E_1^n[\nu]$. The price dampening effect increases (C_g decreases) in the magnitude of disagreement τ_H/τ_L .

For example, even if all traders expect the fundamental value to be \$30 at $t = 3$, each of them also expects others to be interpreting new information incorrectly and thus expects them to revise expectations upon receiving new signal. If prices were equal to the consensus fundamental valuation of \$30, all traders would expect prices to fall in the short run and thus would trade on short-term profit opportunities by "selling" today to buy at better (lower) prices in the next round. The internalization of future disagreement dampens the price today by driving it below \$30, even when all traders correctly agree that \$30 is currently an accurate estimate of fundamental value.

There is also a static dampening effect. The traders' average valuation, $\frac{1}{N} \sum_{n=1}^N E_1^n[\nu]$, differs from what it would be in a model with a common prior where the same amount of information $\tau_H + (N-1)\tau_L$ were known to be split equally across traders. Each trader believes that signals of others have lower precision, and this disagreement pushes down the weight on each private signal in the average valuation $\frac{1}{N} \sum_{n=1}^N E_1^n[\nu]$. In Section 2.3, we show that this effect is implied

⁴Figures 1, 2, and 3 only depend on parameter values of τ_H , τ_L , and N . In all three figures, the total precision is fixed, with $\tau_H + (N-1)\tau_L = 4$ and $N = 100$.

by Jensen's inequality.

In online Appendix B.1, we show that static dampening of private signals in the averages of expectations shows up in an analogous one-period model; it is unrelated to dynamic trading. In online Appendix B.7, we develop a model where static dampening does not occur, but yet prices are dampened and returns exhibit momentum patterns due to $C_g < 1$.

1.4 Return Autocorrelations

We study return predictability in this subsection. We first show that each trader believes that returns exhibit momentum. We then show that there is also positive return autocorrelation under the assumption that empirically correct parameters give all private signals the same precision.

Traders use models with correct structure but with possibly incorrect parameters. While traders' possibly incorrect parameters affect prices, the properties of returns dynamics, such as autocorrelations at different horizons, also depend on empirically correct model specification of private information. Therefore, the properties of return dynamics are functions of *both* traders' (subjective) parameters *and* empirically correct (objective) parameters.

Proposition 2. (1) *Each trader believes that prices exhibit momentum, defined as the covariance of price changes being positive:*

$$\text{Cov}^n[P_2 - P_1, P_1 - P_0] > 0 \quad \text{and} \quad \text{Cov}^n[v - P_2, P_2 - P_1] > 0.$$

(2) *If traders are "correct on average" in the sense that the empirically correct precision of each trader's signal is $\hat{\tau}_n := (\tau_H + (N-1)\tau_L)/N$, then returns exhibit momentum in both periods:*

$$\text{Cov}[P_2 - P_1, P_1 - P_0] > 0 \quad \text{and} \quad \text{Cov}[v - P_2, P_2 - P_1] > 0.$$

Proof: The proof is in Appendix A.4. Since traders are ex ante identical and the prior mean of v is zero, it follows that $P_0 = 0$. Equations (A-5) and (A-6) present closed-form expressions for return covariances. $\square$

Positive return auto-covariance comes from the dampening factor C_g in the equation for price P_1 satisfying $0 < C_g < 1$, not $C_g = 1$.

The equilibrium prices and trading strategies are determined by traders' expectations based on their beliefs. Prices are the same functions of signals i_{nt} under *both* traders' (subjective)

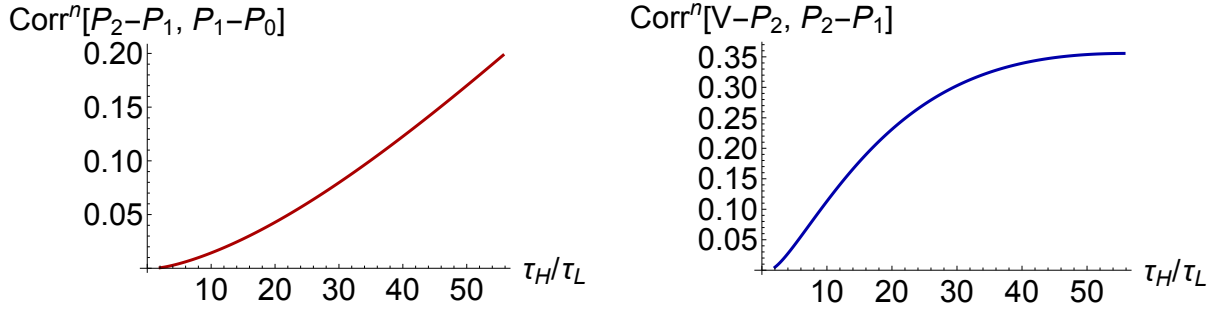


Figure 2: Autocorrelations of Returns, $\text{Corr}^n[P_2 - P_1, P_1 - P_0]$ for different disagreement level τ_H/τ_L .

parameters *and* empirically correct (objective) parameters:

$$\begin{aligned}
 P_1 &= C_g \frac{1}{N} \sum_{n=1}^N E_1^n[\nu] = C_g \frac{\tau_v^{1/2}}{\tau_1} \left(\tau_H^{1/2} + (N-1)\tau_L^{1/2} \right) \frac{1}{N} \sum_{i=1}^n i_{n1}, \\
 P_2 &= \frac{1}{N} \sum_{n=1}^N E_2^n[\nu] = \frac{\tau_v^{1/2}}{\tau_2} \left(\tau_H^{1/2} + (N-1)\tau_L^{1/2} \right) \left(\frac{1}{N} \sum_{i=1}^n i_{n1} + \frac{1}{N} \sum_{i=1}^n i_{n2} \right).
 \end{aligned} \tag{16}$$

Traders' distributions for signals are different from the objective distribution of signals. On the one hand, each trader n believes that his own signal has a high precision $\tau_n = \tau_H$, and other traders' signals have low precision $\tau_m = \tau_L$ for $m \neq n$. Therefore, for traders, the average of private signals is

$$\frac{1}{N} \sum_{n=1}^N i_{nt} = \frac{\tau_H^{1/2} + (N-1)\tau_L^{1/2}}{N} \tau_v^{1/2} \nu + \frac{1}{N} \sum_{n=1}^N e_{nt}. \tag{17}$$

On the other hand, under the empirically correct parameters that all private signals have the same precision, the average of private signals under the empirically correct belief is

$$\frac{1}{N} \sum_{n=1}^N \hat{i}_{nt} = \left(\frac{\tau_H + (N-1)\tau_L}{N} \right)^{1/2} \tau_v^{1/2} \nu + \frac{1}{N} \sum_{n=1}^N e_{nt}. \tag{18}$$

Substituting equation (17) into equation (16), we show analytically that both $\text{Cov}^n[\nu - P_2, P_2 - P_1]$ and $\text{Cov}^n[P_2 - P_1, P_1 - P_0]$ are positive. Substituting equation (18) into equation (16), we show analytically that, under empirically correct parameters which give all private signals the same precision, return covariance $\text{Cov}[\nu - P_2, P_2 - P_1]$ and $\text{Cov}[P_2 - P_1, P_1 - P_0]$ are positive.

Figures 2 and 3 depict how return autocorrelation changes with the disagreement level τ_H/τ_L

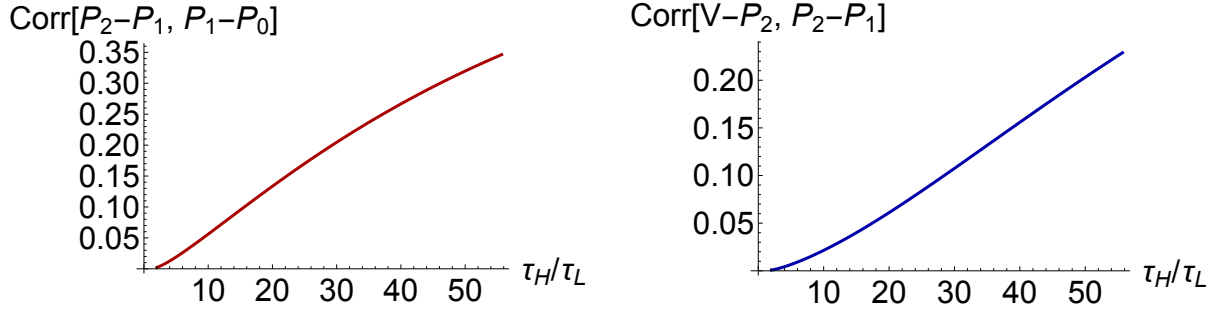


Figure 3: Autocorrelations of Returns, $\text{Corr}[P_2 - P_1, P_1 - P_0]$ under empirically correct parameters for different disagreement level τ_H/τ_L .

while holding fixed the total precision of the signals. Since the price dampening effect increases (C_g decreases) in the magnitude of disagreement τ_H/τ_L , return autocorrelation tends to increase in the disagreement level, under both trader's beliefs and the empirically correct beliefs, as illustrated in Figures 2 and 3.

1.5 Iterated Average Expectations

In this subsection, we discuss how price dampening is related to iterating average expectations. Let $\bar{E}_t$ denote the average expectations operator $\bar{E}_t[\cdot] := \frac{1}{N} \sum_{n=1}^N E_t^n[\cdot]$. We will say that “average expectations iterate” if $\bar{E}_1[\bar{E}_2[\nu]] = \bar{E}_1[\nu]$.

With myopic mean-variance optimizers, equation (4) in Theorem 1 implies that average expectations do not iterate when there is price dampening ($\check{C}_g < 1$):

$$\check{P}_1 = \frac{1}{N} \sum_{n=1}^N E_1^n[\check{P}_2] = \bar{E}_1[\bar{E}_2[\nu]] = \check{C}_g \bar{E}_1[\nu] < \bar{E}_1[\nu]. \quad (19)$$

In the general two-period model, equation (15) similarly implies that average expectations do not iterate when there is price dampening ($C_g < 1$):

$$P_1 = \bar{E}_1[\bar{E}_2[\nu]] - \frac{\text{Cov}_1^n[P_2, E_2^n[\nu] - P_2]}{\text{Var}_1^n[E_2^n[\nu] - P_2] + \text{Var}_2^n[\nu]} (\bar{E}_1[\nu] - \bar{E}_1[\bar{E}_2[\nu]]) = C_g \bar{E}_1[\nu] < \bar{E}_1[\nu]. \quad (20)$$

For expectations to iterate in equations (19) and (20), it must be the case that $\check{C}_g = C_g = 1$ and thus $\check{P}_1 = P_1 = \bar{E}_1[\nu]$. This requires $\tau_H = \tau_L$, which implies that traders share a common prior and also share common information because each trader infers sufficient statistics for other

traders' information from prices. Allen, Morris, and Shin (2006) find that average expectations do not iterate when traders condition on different noisy signals. In their noisy rational expectations model, returns exhibit mean reversion, not momentum.

In models without a common prior, the following proposition provides conditions under which average expectations do iterate.

Proposition 3. (1) *If traders agree about the conditional distribution of v given signals, then $E_1^n[E_2^m[v]] = E_1^n[v]$ for any m and n and thus average expectations iterate, $\bar{E}_1[\bar{E}_2[v]] = \bar{E}_1[v]$.*
 (2) *If traders agree about the joint distribution of signals, then $E_1^n[E_2^m[v]] = E_1^m[v]$ for any m and n and thus average expectations iterate, $\bar{E}_1[\bar{E}_2[v]] = \bar{E}_1[v]$.*
 (3) *If traders agree about the conditional distribution of v given signals or they agree about the joint distribution of signals, then we have $P_1 = E_1^n[P_2]$ for any n , and thus there is no return predictability.*

Proof: The proof is in Appendix A.5. □

Proposition 3 implies that if traders agree about the conditional distribution of v given signals or if traders agree about the joint distribution of signals, then we have $P_1 = E_1^n[P_2]$, implying that all traders believe that price follows a martingale and there is no return predictability. Proposition 3 is similar to Proposition 2 of Banerjee, Kaniel, and Kremer (2009) that, for heterogeneous beliefs to generate price drift, it is necessary (not sufficient) for traders to disagree not only about the joint distribution of signals and fundamental value but also about the joint distribution of signals. They present a model (their Example 3) in which traders receive signals only at date 0 and the aggregate asset supply follows a random walk. Kyle and Wang (2020) show that, although their model has belief heterogeneity, it does not generate price drift; unlike our model, there is no return predictability in their example.

In general, when expectations do not iterate, there may or may not be price dampening and drift. In online Appendix B.6, we construct a model where the average of expectations do not iterate, but under some conditions there is no price dampening or price drift.

2 A Dynamic Model

We next describe a competitive model in which information arrives continuously but trading takes place at discrete intervals. The dynamic steady-state model can be calibrated to generate

quantitatively realistic empirical patterns of return dynamics. In addition, extending to a dynamic trading model also allows us to show that return momentum is more pronounced when traders trade more frequently.

The price aggregates traders' heterogeneous beliefs and private information. Given their beliefs, traders behave in a rational manner. They collect public and private information, construct signals from available information, apply Bayes Law to predict returns, and calculate optimal holdings. Yet, traders are collectively irrational in that each of them is relatively overconfident, believing that the precision of his own private information flow is greater than other traders believe it to be.

2.1 Model Setup

Both fundamentals and information evolve continuously over the time interval $t \in (-\infty, \infty)$. Trading takes place at discrete dates $t = kh$, where $h > 0$ is the time interval between each round of trading, and k indexes time periods $k = \dots, -2, -1, 0, 1, 2, \dots$. Varying h allows us to examine how trading frequency affects equilibrium prices and quantities, with continuous information flows remaining the same. There are N risk-averse perfect competitors who trade a risky asset against a risk-free asset at price P_k at times $t = kh$. The risky security is in zero net supply, and the risk-free asset earns constant risk-free rate $r > 0$.

The risky asset pays dividends at continuous rate $D(t)$. Dividends follow a stochastic process with mean-reverting stochastic growth rate $G^*(t)$, constant instantaneous volatility $\sigma_D > 0$, and constant rate of mean reversion $\alpha_D > 0$:

$$dD(t) := -\alpha_D D(t) dt + G^*(t) dt + \sigma_D dB_D(t). \quad (21)$$

The growth rate $G^*(t)$ follows an AR-1 process with mean-reversion α_G and volatility σ_G :

$$dG^*(t) := -\alpha_G G^*(t) dt + \sigma_G dB_G(t). \quad (22)$$

The dividend stream $D(t)$ is publicly observable. The growth rate $G^*(t)$, marked with a star superscript, is not observed by traders. Each trader uses available information to estimate its value. The model parameters α_D , σ_D , α_G , and σ_G are all common knowledge among traders.

If the dividend $D(t)$ and the growth rate $G^*(t)$ were observable, then the price would equal

its fundamental value from a generalization of Gordon's growth formula:

$$F(t) = \frac{D(t)}{r + \alpha_D} + \frac{G^*(t)}{(r + \alpha_D)(r + \alpha_G)}. \quad (23)$$

Since traders do not observe $G^*(t)$, their valuations have $G^*(t)$ replaced with an estimate based on available information.

Each trader n observes a continuous stream of private information $dI_n(t)$, with precision τ_n , about the unobservable, growth rate $G^*(t)$:

$$dI_n(t) := \tau_n^{1/2} \frac{G^*(t)}{\sigma_G \Omega^{1/2}} dt + dB_n(t). \quad (24)$$

Each increment $dI_n(t)$ in equation (24) is a noisy observation of the unobserved scaled growth rate $G^*(t)$. The scaling factor $\Omega^{1/2}$ is defined in equation (26) below so that τ_n can be interpreted as the rate at which traders learn from signal $dI_n(t)$. Since our model is stationary, Ω is constant. Since traders learn from prices which reveal other traders' signals, the definition of Ω depends on how traders learn from both signals and prices.⁵

The symmetry of the model implies that traders infer from the equilibrium price the average of other traders' information flow about the growth rate:

$$dI_{-n}(t) := \frac{1}{N-1} \sum_{m=1, m \neq n}^N dI_m(t).$$

Each trader n is certain that his own private information $I_n(t)$ has high precision $\tau_n = \tau_H$, and the other traders' private information has low precision $\tau_m = \tau_L$ for $m \neq n$, with $\tau_H > \tau_L \geq 0$.⁶

⁵In addition to scaling the drift term by σ_G , we also scale it by $\Omega^{1/2}$. The scaling is chosen so that linear regression of $G^*(t)$ on $dI_n(t)$ has constant incremental instantaneous R^2 equal to $\tau_n dt$. Traders learn from their own private signal at a constant rate τ_n , which measures the percentage rate at which new information reduces variance, reflected in equation (35) below. Without this scaling, if our model were nonstationary, traders would learn more about $G^*(t)$ in markets where the prior variance of $G^*(t)$ is large and less in the markets where it is small.

⁶A typical way of modeling private information, $dI_n(t) = G^*(t) dt + \tau_n^{-1/2} dB_n(t)$, would multiply the diffusion term $dB_n(t)$ by $\tau_n^{-1/2}$, which measures both the standard deviation of noise and the precision of the signal. Our alternative approach in equation (24) instead multiplies the drift term by its reciprocal $\tau_n^{1/2}$. Rescaling signals by the factor $\tau_n^{1/2}$ does not affect the equilibrium when traders agree about $\tau_n^{1/2}$ but does affect the equilibrium when traders disagree about its value. Using the typical approach in continuous time, a trader can estimate the diffusion variance, and therefore τ_n , with high accuracy by measuring the quadratic variation of $dI_n(t)$ over short time intervals. This would conflict with our assumption that traders always disagree about the precision of their private signals. The information structure defined in equation (24) has the appealing feature that traders infer the correct

Since this disagreement is a common knowledge, relatively overconfident traders agree to disagree about the precisions of their signals. This disagreement generates trading. Traders believe that they can make profits at the expense of others, even though it is a common knowledge that aggregate profits are equal to zero.

Each trader also infers some information about the growth rate $G^*(t)$ from the publicly observable dividend stream $D(t)$ modelled in equation (21). To streamline notation for the information content of dividends, define $dI_0(t) := (\alpha_D D(t) dt + dD(t))/\sigma_D$ with $dB_0 := dB_D$ and

$$\tau_0 := \Omega \sigma_G^2 / \sigma_D^2.$$

Then, the stochastic process

$$dI_0(t) := \tau_0^{1/2} \frac{G^*(t)}{\sigma_G \Omega^{1/2}} dt + dB_0(t) \quad (25)$$

is informationally equivalent to the dividend process $D(t)$ in equation (21). Assume it is common knowledge that the Brownian motions $dB_0, dB_G, dB_1, \dots, dB_N$ are independently distributed. Traders agree on the precision τ_0 of public information in equation (25).

Since each trader believes his own signal has high precision τ_H and others' signals have low precision τ_L , the symmetry implies that traders agree on the total precision τ of information flows in the model:

$$\tau := \tau_0 + \tau_H + (N-1)\tau_L.$$

Let $E_{kh}^n[\dots]$ denote the expectation of trader n calculated with respect to his beliefs about parameter values using information at time $t = kh$. This information consists of the history of continuous flows of public signals $dI_0(j)$, continuous flow of private signals $dI_n(j)$ with $j \in [-\infty, t]$, and discrete sequence of prices P_j with $j \leq k$. Let

$$G_n(t) := E_t^n[G^*(t)] \quad \text{and} \quad G_{n,k} := E_{kh}^n[G^*(kh)]$$

denote trader n 's estimate of the growth rate at time t and discrete times $t = kh$, respectively. We will use similar notation for other continuous and discrete variables.

Define Ω as the steady state error variance of trader n 's estimate of $G^*(t)$, scaled in units of

diffusion variance $\text{Var}[dB_n(t)] = dt$. By assuming $\tau_n^{1/2}$ multiplies the drift term in equation (24), traders disagree about how to interpret parameters of the drift terms, which are much more difficult to estimate empirically.

the standard deviation of its innovation σ_G :

$$\Omega := \text{Var} \left[\frac{G^*(t) - G_n(t)}{\sigma_G} \right]. \quad (26)$$

The variable Ω has no time subscript k because it is constant in our steady-state model, where the total precision of traders' private information flow is fixed. Except for symmetrically disagreeing about the precisions τ_H and τ_L of signals, traders agree about all parameter values. Since symmetry implies that all traders agree about the variance Ω , it has no subscript n .

Each trader chooses optimal consumption and portfolio holdings to maximize an additively separable exponential utility function with risk aversion A and time preference ρ :

$$\mathbb{E}_{kh}^n \left[\int_{t=kh}^{\infty} e^{-\rho(t-kh)} U(c_n(t)) dt \right]. \quad (27)$$

The optimization problem is complicated by the fact that consumption $c_n(t)$ is chosen continuously while portfolio holdings $S_{n,k}$ change only at discrete trading times $t = kh$. For analytical tractability, we simplify the problem slightly by assuming that when trading occurs at round j , each trader chooses both portfolio holdings $S_{n,j}$ and a consumption budget $hc_{n,j}$, which do not change until the next trading round. Thus, traders do not use new public or private information unfolding between trading rounds to adjust consumption between rounds. They cannot use other traders' private information between trading rounds either, because there are no updated prices from which to infer the average of other traders' signals.

With these assumptions, the optimization problem becomes the discrete-time problem of choosing consumption budget $hc_{n,j}$ and holdings $S_{n,j}$ at trading times $t = jh$, to solve

$$\max_{\substack{[c_{n,j}], j=k, k+1, \dots, \infty \\ [S_{n,j}], j=k, k+1, \dots, \infty}} \mathbb{E}_{kh}^n \left[\sum_{j=k}^{\infty} e^{-\rho(j-k)h} U_{n,j}(c_{n,j}) \right], \quad (28)$$

subject to the budget constraint at each trading round $j + 1$,

$$W_{n,j+1} = e^{rh} (W_{n,j} - hc_{n,j} - S_{n,j}P_j) + S_{n,j}\check{D}_{j+1} + S_{n,j}P_{j+1}, \quad (29)$$

where $\check{D}_{j+1}$ defines the future value of dividends between trading rounds j and $j+1$,

$$\check{D}_{j+1} = e^{rh} \int_{jh}^{(j+1)h} e^{-r(t-jh)} D(t) dt, \quad (30)$$

and $U_{n,j}(c_{nj})$ solves the nested continuous-time consumption subproblem between trading rounds j and $j+1$,

$$U_{n,j}(c_{nj}) := \max_{c_n(\cdot)} E_{jh}^n \left[- \int_0^h e^{-\rho u} e^{-Ac_n(jh+u)} du \right], \quad (31)$$

subject to consumption budget $hc_{n,j}$ allocated for the time period between rounds j and $j+1$,

$$hc_{n,j} = \int_0^h e^{-ru} c_n(jh+u) du. \quad (32)$$

Equations (28)–(32) summarize the optimization problem.

As we prove in Lemma 1 in Appendix A.6, Stratonovich–Kalman–Bucy filtering implies that trader n 's estimate $G_{n,k}$ of the growth rate at period k can be conveniently written as the weighted sum of the three sufficient statistics $H_{0,k}$, $H_{n,k}$, and $H_{-n,k}$, which summarize the information content of dividends, trader n 's private information, and other traders' private information, respectively, with

$$H_{n,k} := \int_{t=-\infty}^{kh} e^{-(\alpha_G + \tau)(kh-t)} dI_n(t), \quad n = 0, 1, \dots, N, \quad (33)$$

and

$$H_{-n,k} := \frac{1}{N-1} \sum_{m=1, m \neq n}^N H_{m,k}, \quad n = 1, \dots, N. \quad (34)$$

These formulas for sufficient statistics have an intuitive interpretation. The signal $H_{n,k}$ is a sufficient statistic for trader n 's own information flow. The average signal $H_{-n,k}$ is a sufficient statistic for other traders' information flow. The importance of each bit of information dI_n decays exponentially at a rate $\alpha_G + \tau$, the sum of the natural decay rate α_G of the growth rate and the speed τ at which traders learn about it. Although trader n does not observe other traders' private signals directly, the sufficient statistic $H_{-n,k}$ can be inferred from prices.

This filtering also implies that the steady-state error variance is

$$\Omega = \text{Var} \left[\frac{G^*(t) - G_n(t)}{\sigma_G} \right] = \frac{1}{2\alpha_G + \tau}, \quad (35)$$

and trader n 's expected growth rate at $t = kh$ is

$$G_{n,k} := E_{kh}^n [G^*(kh)] = \sigma_G \Omega^{1/2} \left(\tau_0^{1/2} H_{0,k} + \tau_H^{1/2} H_{n,k} + (N-1) \tau_L^{1/2} H_{-n,k} \right). \quad (36)$$

When forming his estimate, trader n assigns a weight $\tau_0^{1/2}$ to the public signal $H_{0,k}$, a weight $\tau_H^{1/2}$ to his own signal $H_{n,k}$ and a smaller weight $\tau_L^{1/2}$ to each of the other traders' signals $H_{m,k}$, for $m \neq n$. Trade occurs as a result of the symmetrically different weights used by traders in construction of their estimates.

Define the aggregate sufficient statistic H_k at period k , with common weights $\tau_I^{1/2}$, as

$$H_k := \tau_0^{1/2} H_{0,k} + \sum_{n=1}^N \tau_I^{1/2} H_{n,k}, \quad \text{where} \quad \tau_I^{1/2} := \frac{1}{N} \tau_H^{1/2} + \frac{N-1}{N} \tau_L^{1/2}. \quad (37)$$

Then, the average estimate of the growth rate $\bar{G}_k$ in period k is proportional to H_k :

$$\bar{G}_k := \frac{1}{N} \sum_{n=1}^N G_{n,k} = \sigma_G \Omega^{1/2} H_k. \quad (38)$$

2.2 The Equilibrium with Myopic Conditional Mean-Variance Optimizers

Similar to the two-period model, we first solve for the equilibrium prices where traders are myopic and hold conditional mean-variance optimal portfolio:

$$\check{S}_{n,k} = \frac{E_{kh}^n [\check{P}_{k+1} + \check{D}_{k+1}] - e^{rh} \check{P}_k}{A \text{Var}_{kh}^n [\check{P}_{k+1} + \check{D}_{k+1}]}. \quad (39)$$

The equilibrium price $\check{P}_k$ is given in the following theorem.

Theorem 3 (Prices with Myopic Conditional Mean-Variance Optimizers). *Suppose each trader holds conditional mean-variance optimal portfolio at each period, then the price $\check{P}_k$ is*

$$\check{P}_k = \frac{D_k}{r + \alpha_D} + \check{C}_G \frac{\bar{G}_k}{(r + \alpha_D)(r + \alpha_G)}, \quad (40)$$

where $\bar{G}_k$, defined in equation (38), is the average of traders' expected growth rates at time kh .

The coefficient $\check{C}_G$ satisfies $0 < \check{C}_G < 1$ and is given as

$$\check{C}_G := \left(1 + \frac{1 - e^{-\tau h}}{e^{(r+\alpha_G)h} - 1} \frac{N-1}{N} \left(\tau_H^{1/2} - \tau_L^{1/2} \right)^2 / \tau \right)^{-1}. \quad (41)$$

Proof: The proof of Theorem 3 is presented in Appendix A.7. $\square$

The pricing formula (40) resembles an average of traders' valuations of fundamentals (23), with one important difference. The coefficient $\check{C}_G$ satisfies $0 < \check{C}_G < 1$. This dampens prices relative to the traders' average valuation. The endogenous parameter $\check{C}_G$ measures price dampening just like the coefficient $\check{C}_g < 1$ and $C_g < 1$ in the two-period model in Section 1.

Proposition 4. *A common prior ($\tau_H = \tau_L$) implies no price dampening, with $\check{C}_G = 1$. Relative overconfidence ($\tau_H > \tau_L$) implies $0 < \check{C}_G < 1$. The price dampening effect $\check{C}_G$ increases in the trading interval h and decreases in the disagreement τ_H/τ_L (holding the total precision τ fixed).*

Proof: The proof of Proposition 4 is in Appendix A.8. $\square$

Proposition 4 implies that price dampening becomes more pronounced with more frequent trading. When h becomes smaller, traders exploit profitable trading opportunities more frequently. If $h \rightarrow \infty$, then $\hat{C}_G = 1$. This implies that if traders can only buy and hold the risky asset, then the price is exactly equal to traders' average valuations of fundamentals. The dampening effect $\check{C}_G < 1$ occurs because traders internalize their future disagreement about the growth rate. Each trader agrees to disagree with others about how to interpret private signals at the present and how to interpret private signals in the future. When internalized, this future disagreement dampens the market price at the present relative to average valuation. Each trader expects that as time passes, other traders will revise their mistaken current valuations of fundamentals towards unconditional levels, before they ultimately in the long run converge towards his own "correct" valuation. He tries to profit by trading ahead of anticipated short-term revisions of others' expectations, even if this means trading against his own long-term valuation. This short-term speculative trading dampens current prices relative to the average of traders' valuations. In equilibrium, this leads to a Keynesian beauty contest. Yet, contrary to the common intuition, this short-term speculative trading tends to dampen, rather than magnify, price fluctuations.

We next discuss trader n 's expectations about his and others' stock positions in the next period. Suppose trader n thinks that he can sell the stock at time $(k+1)h$ at price equal to his own

valuation of fundamentals, $F_{n,k+1} = \frac{D_{k+1}}{r+\alpha_D} + \frac{G_{n,k+1}}{(r+\alpha_D)(r+\alpha_G)}$, then his conditional mean-variance optimal portfolio at time kh is given as in equation (39), replacing the price $\check{P}_{k+1}$ with $F_{n,k+1}$. It can be shown that the price at time kh , $\check{P}_k = \frac{D_k}{r+\alpha_D} + \frac{\bar{G}_k}{(r+\alpha_D)(r+\alpha_G)}$, and, at time $(k-1)h$, trader n expects his conditional mean-variance position at time kh , $E_{(k-1)h}^n[\check{S}_{n,k}]$, is proportional to

$$E_{(k-1)h}^n[G_{n,k} - G_{-n,k}] = e^{-\alpha_G h} \left(\frac{1}{\tau} \left(\tau_H^{1/2} - \tau_L^{1/2} \right)^2 \left(1 - e^{-\tau h} \right) G_{n,k-1} + e^{-\tau h} (G_{n,k-1} - G_{-n,k-1}) \right). \quad (42)$$

If all traders happen to agree about the fundamental valuation at time $(k-1)h$ (i.e., $G_{n,k-1} = G_{-n,k-1} > 0$), then Equation (42) is reduced to

$$E_{(k-1)h}^n[G_{n,k} - G_{-n,k}] = \frac{1}{\tau} \left(\tau_H^{1/2} - \tau_L^{1/2} \right)^2 e^{-\alpha_G h} (1 - e^{-\tau h}) G_{n,k-1} > 0. \quad (43)$$

Each trader n expects himself to buy ($E_{(k-1)h}^n[\check{S}_{n,k}] > 0$) and others to sell next period because each trader expects prices to fall in the next period. Each trader believes that others make the mistake of attributing too much precision to their current and future signals. Each trader anticipates that, upon observing new information in the future, others will adjust their valuations towards unconditional levels, and thus each trader expects others to sell and prices to fall.

Trader n 's conditional mean-variance optimal portfolio, $\check{S}_{n,k-1}$ is proportional to

$$\begin{aligned} E_{(k-1)h}^n \left[e^{-rh} \left(\frac{D_k}{r+\alpha_D} + \frac{\bar{G}_k}{(r+\alpha_D)(r+\alpha_G)} + \check{D}_{k+1} \right) \right] - \check{P}_{k-1} \\ = \frac{D_{k-1}}{r+\alpha_D} + \frac{G_{n,k-1}}{(r+\alpha_D)(r+\alpha_G)} - \frac{N-1}{N} e^{-rh} \frac{E_{k-1}^n[G_{n,k} - G_{-n,k}]}{(r+\alpha_D)(r+\alpha_G)} - \check{P}_{k-1}. \end{aligned} \quad (44)$$

Equations (43) and (44) imply that, if it happens that traders agree about the fundamental valuation at time $(k-1)h$ (so $G_{n,k-1} = G_{-n,k-1} > 0$) and the price $\check{P}_{k-1}$ is equal to their consensus fundamental valuation, $\check{P}_{k-1} = \frac{D_{k-1}}{r+\alpha_D} + \frac{\bar{G}_{k-1}}{(r+\alpha_D)(r+\alpha_G)}$, then all traders would want to hold short positions at time $(k-1)h$ because all of them would expect prices to fall below fundamental value at time kh . In equilibrium, this short-term speculative trading dampens current prices relative to the average of traders' valuations. The price dampening becomes more pronounced when traders exploit profitable trading opportunities more frequently.

2.3 The Equilibrium of the Dynamic Trading Model

We next solve for the dynamic trading model where trader n chooses consumption and holdings to solve his optimal optimization problem (27). Each trader's optimal inventory is proportional to his own risk tolerance and the difference between his valuation and the average valuation of other traders, which he infers from prices. The following theorem characterizes equilibrium for the discrete-time dynamic model.

Theorem 4 (Equilibrium of the Dynamic Trading Model). *There exists a steady-state competitive equilibrium with symmetric linear strategies and with positive trading volume if and only if the three polynomial equations (A-33)–(A-34) have a solution, and traders' demand curves are downward sloping. Such an equilibrium has the following properties:*

1. *There is an endogenously determined constant $C_L > 0$, defined in equation (A-30), such that trader n 's optimal inventories $S_{n,k}$ at period k are*

$$S_{n,k} = C_L (H_{n,k} - H_{-n,k}). \quad (45)$$

2. *There is an endogenously determined constant $C_G > 0$, defined in equation (A-28), such that the equilibrium price at period k is*

$$P_k = \frac{D_k}{r + \alpha_D} + C_G \frac{\bar{G}_k}{(r + \alpha_D)(r + \alpha_G)}. \quad (46)$$

Proof: The dynamics of main state variables $H_{n,k}$ and $H_{-n,k}$ and the proof of Theorem 4 are presented in Appendix A.7. □

Theorem 4 implies that competitive traders immediately adjust inventories to optimal levels; they do not smooth their trading out over time to reduce price impact. With relative overconfidence ($\tau_H > \tau_L$), extensive numerical investigation suggests $C_G < 1$ in the general dynamic model. In the limit $h \rightarrow 0$, we prove $C_G < 1$ when traders are relatively overconfident ($\tau_H > \tau_L$). Similar to the case with myopic conditional mean-variance optimizers, the pricing formula (46) implies that the price is dampened relative to the traders' average valuation.

We derive the stochastic processes under trader n 's information set for estimates $G_n(t)$ and

$G_{-n}(t) := \frac{1}{N-1} \sum_{m=1, m \neq n}^N G_m(t)$ as: $dG_n(t) = -\alpha_G G_n(t) dt + \text{“dB-terms”}$, and

$$dG_{-n}(t) = -\left(\alpha_G + \left(\tau_H^{1/2} - \tau_L^{1/2}\right)^2\right) G_{-n}(t) dt + \left(\tau_0 + \tau_L^{1/2} \left(2\tau_H^{1/2} + (N-2)\tau_L^{1/2}\right)\right) (G_n(t) - G_{-n}(t)) dt + \text{“dB-terms”}. \quad (47)$$

Trader n believes that his own growth rate estimate $G_n(t)$ will mean-revert to zero at a rate α_G ; his expectations satisfy the law of iterated expectations. Trader n also believes other traders' growth rate estimates $G_{-n}(t)$ will follow a more complicated path: (1) they will mean-revert to zero at a rate $\alpha_G + \left(\tau_H^{1/2} - \tau_L^{1/2}\right)^2 > \alpha_G$ and (2) they will also tend to drift towards trader n 's estimate $G_n(t)$. If $G_n(t) = G_{-n}(t)$, then trader n believes that other traders' estimates $G_{-n}(t)$ will mean-revert to zero at a rate $\alpha_G + \left(\tau_H^{1/2} - \tau_L^{1/2}\right)^2 > \alpha_G$.

Figure 4 further illustrates the intuition behind the dynamic price dampening effect. For simplicity, we assume that at $t = 0$ all traders have the same buy-and-hold valuations of \$30.⁷ Yet, the price is lower than the average of their valuations due to the dampening effect. The figure shows trader n 's hypothetical expected buy-and-hold valuation at $t = 0$ under the assumption that he buys the asset at $t = 0$, holds it until time $T = fh$, and then sells it at some hypothetical price. While calculations are made under trader n 's beliefs, they are symmetrically identical for all traders. Different buy-and-hold valuations $PV_{n,f}$, $PV_{-n,f}$, and $PV_{p,f}$ on the vertical axis are plotted against various holding horizons $T = fh$ on the horizontal axis (for trading frequencies $h = 0.1, 1, 2$, and 5). A formal analysis of expectations dynamics is provided in Appendix B.2.

The thick solid horizontal line $PV_{n,f}$ is based on the assumption that trader n holds the risky asset until date $T = fh$ and then sells it for what he expects at $t = 0$ that his valuation of fundamentals will be at time T . Since given his beliefs, trader n correctly applies Bayes law, the martingale property of his valuation (the law of iterated expectations) implies that the present value $PV_{n,f}$ is a constant equal to \$30 for any horizons T .

The thin solid curve $PV_{-n,f}$, just below the line for $PV_{n,f}$, is the present value of the risky asset based on the assumption that trader n holds the asset until time $T = fh$ and then liquidates his position at what he expects at $t = 0$ the average of the other $N - 1$ traders' valuations

⁷The buy-and-hold value of \$30 corresponds to $G_n(0) = G_{-n}(0) = 0.105$, $D(0) = 1$, and parameter values r , A , N , α_D , α_G , σ_D , σ_G , τ_H , and τ_L are given in Table 2 in Section 4. Details of present-value calculations are given in equations (B-6), (B-8), and (B-10) in Appendix B.2.

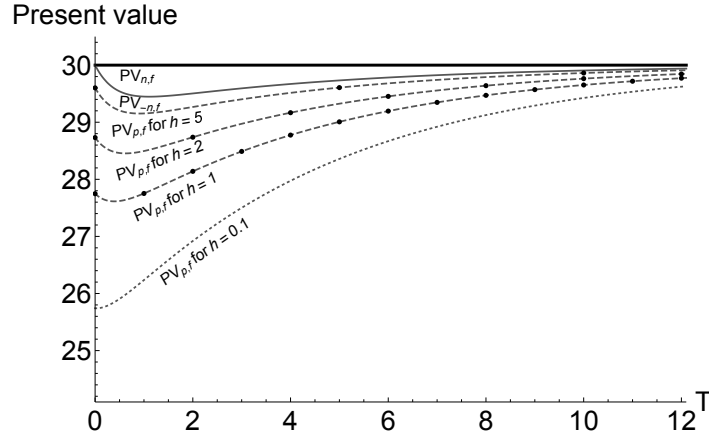


Figure 4: Present Value of Dividends and Liquidation Value from the Perspective of a Trader. The thick (thin) solid curve $PV_{n,f}$ ($PV_{-n,f}$) is the present value when trader n sells the stock at price equal to his own valuation of fundamentals (the average of other traders' valuations). The dashed curves $PV_{p,f}$ are the present values when trader n sells the stock at the market price at time $T = fh$ for trading frequencies $h = 0.1, 1, 2$, and 5 .

of the fundamental value will be at time T . Trader n believes that the other traders' estimates of the growth rate will mean revert to zero at rate $\alpha_G + \left(\tau_H^{1/2} - \tau_L^{1/2}\right)^2$, which is faster than the mean reversion rate α_G he attributes to his own estimate. Thus, $PV_{-n,f}$ will first fall toward its unconditional level (of zero) and then in the long run rise back to catch up with trader n 's own (correct, in his opinion) estimate of the fundamental value equal to \$30. These buy-and-hold valuations deviate from the horizontal line at \$30.

Figure 4 shows that prices are dampened relative to traders' valuations. The four dashed curves $PV_{p,f}$ correspond to the present values of the risky asset when trader n holds the risky asset until time $T = fh$ and then sells it at the market price at time T assuming trading frequencies $h = 0.1, 1, 2$, and 5 . Consistent with the equilibrium result $0 < C_G < 1$, the initial price $P_0 := PV_{p,0}$ is lower than the consensus valuation of \$30, even if all traders agree about this valuation at $t = 0$. Indeed, if prices were equal to the consensus fundamental valuation of \$30, all traders would expect prices to fall in the short run and thus would want to hold short positions. As a result, the price P_0 at $t = 0$ is dampened relative to the average of traders' estimates of fundamental valuations. Traders agree to disagree about the dynamics of their future valuations, and internalization of this future disagreement dampens the market price at the present. Figure 4 also shows that price dampening becomes more pronounced with more frequent trading.

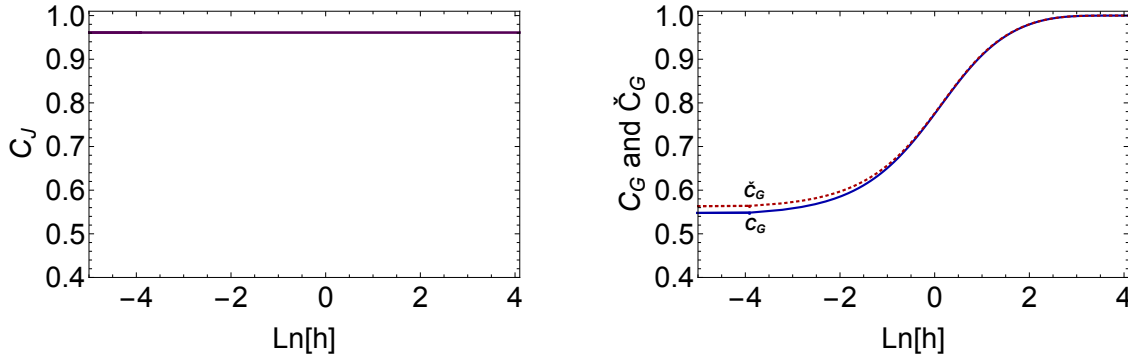


Figure 5: C_J , $\check{C}_G$ (dashed), and C_G (solid) against time interval $\ln[h]$.

When h becomes smaller, traders exploit profitable trading opportunities more frequently. In contrast, if $h \rightarrow \infty$ and traders can only buy and hold the risky asset, then $C_G = 1$ and the price is exactly equal to traders' average valuation of fundamentals.

There is also the second dampening effect. The average valuation without a common prior differs from what it would be if the same precision $\tau_H + (N-1)\tau_L$ were known to be split equally across traders. The average of traders' estimates of the growth rate $\bar{G}_k$, defined in (38) and related to equilibrium prices (46), can be then expressed as

$$\bar{G}_k = \sigma_G \Omega^{1/2} \left(\tau_0^{1/2} H_{0,k} + C_J \left(\frac{1}{N} \tau_H + \frac{N-1}{N} \tau_L \right)^{1/2} \sum_{n=1}^N H_{n,k} \right), \quad \text{where } C_J := \frac{\frac{1}{N} \tau_H^{1/2} + \frac{N-1}{N} \tau_L^{1/2}}{\left(\frac{1}{N} \tau_H + \frac{N-1}{N} \tau_L \right)^{1/2}}. \quad (48)$$

The constant C_J measures the bias in how weights of each private (but not public) signal is dampened in the average valuation $\bar{G}_k$ due to relative overconfidence among traders; it is defined as the ratio of the average of the square roots to the square root of the average of precisions. When traders are relatively overconfident ($\tau_H > \tau_L$), Jensen's inequality implies $C_J < 1$.

Out of two dampening effects, the dynamic price dampening effect of $C_G < 1$ is the main driving force behind price momentum and return predictability.

Figure 5 shows how the constants C_J , $\check{C}_G$, and C_G depend on the time interval h between trading rounds.⁸ The difference between the endogenous parameters $\check{C}_G$ and C_G is small for most parameter values, implying that the dampening effect mainly arises due to traders' short-term speculative trading. The coefficients $\check{C}_G$ and C_G increase with h and ultimately converges

⁸In Figures 5 and 6, parameter values r , A , N , α_D , α_G , σ_D , and σ_G are given in Table 2 in Section 4. In Figure 5, τ_H , and τ_L are given in Table 2 in Section 4, and in Figure 6, $\tau = 2.2$, $h = 0.01, 0.5$, and 2 .

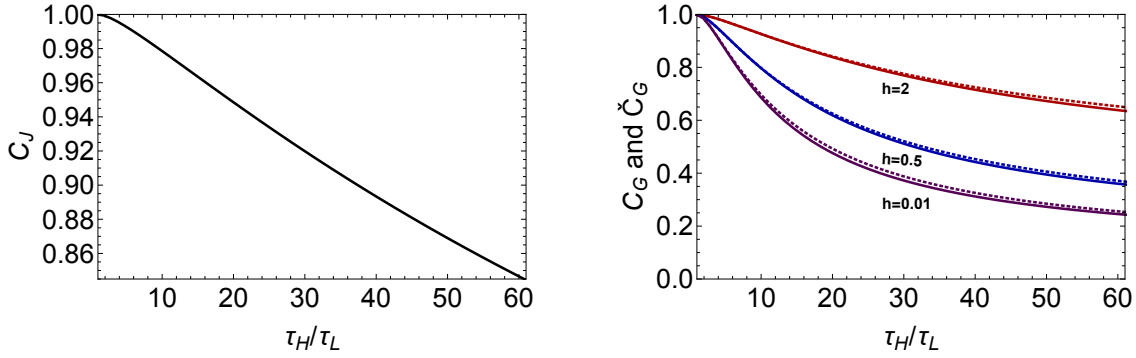


Figure 6: C_J , $\check{C}_G$ (dashed), and C_G (solid) against τ_H/τ_L holding fixed τ .

to one, as trading opportunities occur at less frequent intervals (h gets large) and it gets more difficult for traders to take advantage of short-term opportunities. When h gets closer to zero, traders trade more aggressively against each other's perceived mistakes. The coefficients $\check{C}_G$ and C_G become flat when $\ln(h) < -2$; thus, the results of the discrete-time model converge to those of the continuous-time one approximately when $h < 0.135$ years, corresponding to about seven weeks in this example. As expected from its definition (48), the coefficient C_J does not depend on h .

Figure 6 illustrates how constants C_J , $\check{C}_G$, and C_G depend on the level of disagreement τ_H/τ_L . Holding total precision fixed, C_J , $\check{C}_G$, and C_G gradually decrease as disagreement τ_H/τ_L increases. Both more disagreement and more trading opportunities lead to more pronounced price dampening $C_G < 1$ ($\check{C}_G < 1$) since traders have greater incentives to engage in short-term speculation. Figures 5 and 6 show that more disagreement also amplifies the effect of $C_J < 1$, since it magnifies the effect of Jensen's inequality. The price dampening effect of $C_G < 1$ is usually much greater than the effect of $C_J < 1$.

Proposition 5. *A common prior ($\tau_H = \tau_L$) implies no price dampening, with $C_G = 1$ and $C_J = 1$. Relative overconfidence ($\tau_H > \tau_L$) implies $0 < \lim_{h \rightarrow 0} C_G \leq \lim_{h \rightarrow 0} \check{C}_G < 1$, and $0 < C_J < 1$.*

The proof is in Appendix A.9.

Proposition 6. *The risk tolerance parameter $1/A$ scales trading volume but has no effect on prices or on the constants $\check{C}_G$, C_G , and C_J .*

The proof is in the Appendix A.10. This proposition implies that price dampening arises due to specific features of information processing rather than risk sharing.

3 Return Predictability

We next derive a structural model for return dynamics and examine its time-series properties. Equilibrium strategies and prices depend only on traders' heterogeneous expectations based on inconsistent beliefs about parameter values. Their contradictory beliefs cannot by construction all be empirically correct. To study return dynamics empirically, we introduce a set of empirically “correct” beliefs describing the true data-generating process. Empirical predictions about price dynamics, including the auto-covariance structure of returns, must depend on *both* the possibly incorrect (subjective) parameters used by traders *and* the empirically correct (objective) parameters describing market dynamics, even if no trader actually has correct beliefs.

3.1 Inference under Correct Beliefs

We first introduce empirically correct parameters. Let “hats” distinguish empirically correct parameter values from the possibly incorrect beliefs of traders.

In a symmetric model, it is natural to assume that all signals have the same precision, and thus no trader interprets information flow correctly. Therefore, assume traders believe correctly that total private precision is τ and public precision is τ_0 , but—in contrast to traders' overconfident beliefs that their own signals have precision τ_H and other traders' signals have precision τ_L —all private signals $n = 1, \dots, N$ have symmetrically the same empirically correct precision

$$\hat{\tau}_n = \hat{\tau}_I := \frac{\tau_H + (N-1)\tau_L}{N}. \quad (49)$$

In this section, except for τ_n , $n = 1, \dots, N$, we assume that traders have empirically correct beliefs about all other model parameters.

In Appendix A.11.1, we derive filtering formulas using an approach similar to Section 2.1. The mathematical formulas are different because the analysis distinguishes between the true precision of signals $\hat{\tau}_I$ and traders' incorrect beliefs τ_H and τ_L . The history of each information flow $I_n(t)$ can be summarized by the same sufficient statistic $H_{n,k}$ as in equation (33) because traders use the correct decay rate $\tau + \alpha_G$ to deflate past information ($\hat{H}_{n,k} = H_{n,k}$). We can also define the aggregate sufficient statistic $\hat{H}_k$ as a linear combination of $H_{n,k}$, $n = 0, \dots, N$:

$$\hat{H}_k = \tau_0^{1/2} H_{0,k} + \sum_{n=1}^N \hat{\tau}_I^{1/2} H_{n,k}. \quad (50)$$

The aggregate sufficient statistic $\hat{H}_k$ is defined similarly to H_k in equation (37), but the coefficient of $H_{n,k}$ is, importantly, $\hat{\tau}_I^{1/2}$ in (49) rather than $\tau_I^{1/2}$ in (37).

Let $\hat{E}_{kh}[\dots]$ denote the empirically correct expectation operator given all information at time $t = kh$. Under correct beliefs, the estimate of the steady-state error variance is the same Ω as in equation (35), but the estimate of the growth rate at time $t = kh$ is

$$\hat{G}_k := \hat{E}_{kh}[G^*(kh)] = \sigma_G \Omega^{1/2} \hat{H}_k. \quad (51)$$

The estimate of the growth rate $\hat{G}_k$ is defined like $\bar{G}_k$ in equation (38), but it depends on the aggregate sufficient statistic $\hat{H}_k$ rather than the aggregate sufficient statistic H_k .

Even though equilibrium prices and strategies depend only on traders' expectations based on their inconsistent beliefs, the objective empirical properties of prices also depend on empirically correct parameters. From equation (46), the price P_k at time kh depends on the dividend D_k , traders' sufficient statistics H_k about the growth rate; it does not depend on $\hat{H}_k$. Yet, to construct returns, one must project the dynamics of D_k and H_k into the future by modelling how information flows $dI_n(t)$, $n = 0, 1, \dots, N$, will unfold after time kh ; this projection must be done under correct parameter values for precisions. The true evolution of these processes also depends on the correct forecasts of growth rate $\hat{G}_k$ from equation (51), which themselves depend on the sufficient statistics $\hat{H}_k$, defined in (50), not H_k . Since the model is symmetric, the sufficient statistic $\hat{H}_k$ can be obtained from the traders' statistics $H_{0,k}$ and H_k , which in turn can be inferred from the current dividends and prices.

3.2 Return Autocorrelations

Let $R_{k,k+f}$ denote the cumulative net holding period excess return, measured in dollars per share, from buying one share at trade date $t = kh$, financing the purchase at the risk-free rate, and selling the asset f trades dates later at $t + T = (k + f)h$:

$$R_{k,k+f} = P_{k+f} - e^{rT} P_k + e^{rT} \int_{t=kh}^{kh+T} e^{-r(t-kh)} D(t) dt.$$

The holding-period return sums capital gains and dividends. It can be expressed as a linear combination of current and past dividends and prices.

Traders use models with correct structure but with possibly incorrect parameters. While

traders' possibly incorrect parameters affect prices, the properties of returns dynamics also depend on empirically correct model specification of dividends and information. Therefore, the properties of return dynamics are function of *both* traders' (subjective) parameters *and* empirically correct (objective) parameters.⁹

Theorem 5 (Expected Holding-Period Returns). *Assume $\hat{\alpha}_G = \alpha_G$ and $\hat{\tau} = \tau$, the expected holding-period return over period $T = fh$ from time $t = kh$ to $(k + f)h$ is*

$$\hat{E}_{kh}[R_{k,k+f}] = \beta_1(T) \left(P_k - \frac{D_k}{r + \alpha_D} \right) - \beta_0(T) H_{0,k}. \quad (52)$$

1. *If traders have a common prior ($\tau_H = \tau_L$), then coefficients satisfy $\beta_0(T) = \beta_1(T) = 0$. The price is a martingale, and the expected return is zero.*
2. *If traders are relatively overconfident ($\tau_H > \tau_L$), then coefficients satisfy $\beta_0(T) > 0$, and $\beta_1(T) > 0$ is monotonically increasing in the horizon T .*

The proof is in the Appendix A.11. In the benchmark case with common prior $\tau_H = \tau_L$, expected returns are zero for all horizons. Otherwise, holding-period excess returns $\hat{E}_{kh}[R_{k,k+f}]$ depend on two terms capturing (1) time-series momentum related to the deviation of the current price P_k from its unconditional value $D_k/(r + \alpha_D)$, and (2) overreaction to the history of public information.

The first term in equation (52) is related to price dampening and time-series momentum. In Appendix A.11, we prove that the coefficient $\beta_1(T)$ is positive and it can be decomposed into two terms, where the first term is positive due to the static dampening effect of $C_J < 1$ and the second term is positive due to the dynamic dampening effect of $C_G < 1$. Since $\beta_1(T)$ is positive, future returns are expected to be positive when the current price P_k in (46) is above its unconditional valuation, which suggests positive returns in the past, and future returns are expected to be negative when the price is below its long-term level, which implies negative returns in the past.

The second term is proportional to the public signal. Traders put too much weight on public signals relative to the dampened weights on private signals. This overreaction to public in-

⁹In the online Appendix B.8, we examine return autocorrelation as a function of traders' beliefs. We show that dynamic price dampening leads to time-series return momentum, and momentum is more substantial when traders can trade more frequently.

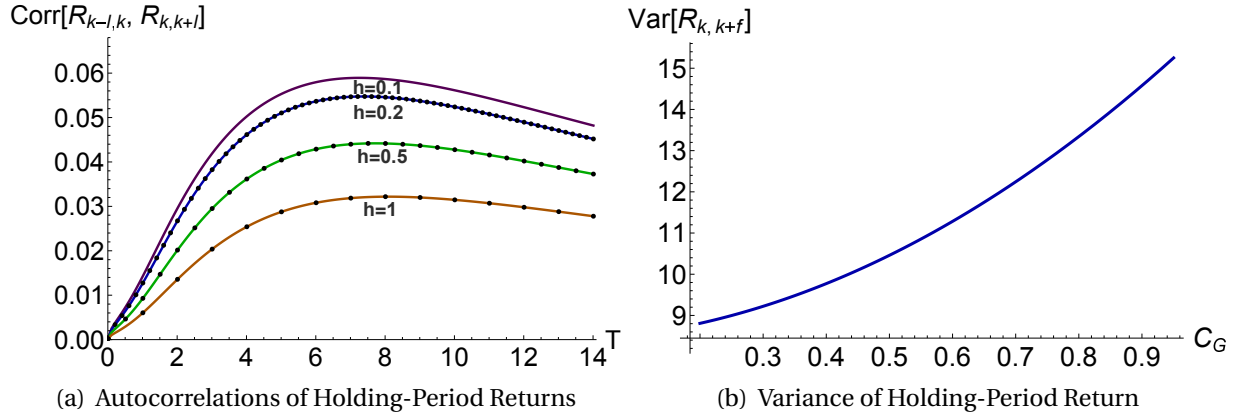


Figure 7: Autocorrelations of Holding-Period Returns, $\text{Corr}[R_{k-l,k}, R_{k,k+l}]$ over holding periods $T = lh$ and Variance of Holding-Period Return, $\text{Var}[R_{k,k+f}]$ against C_G for $T = fh$.

formation $H_{0,k}$, aggregated over a long period of positive or negative news, is expected to be corrected going forward.

We next examine return autocorrelations $\text{Corr}[R_{k-l,k}, R_{k,k+f}]$, assuming for simplicity equal leads and lags $f = l$ or $T = T_l = T_f$. Derivations are in online Appendix B.5. The left panel of Figure 7 depicts return autocorrelations $\text{Corr}[R_{k-l,k}, R_{k,k+l}]$ for horizons $T = lh$; plots are presented for different trading intervals.¹⁰ The return autocorrelations are positive and decreasing with trading frequency h , implying that the short-run momentum is more significant when h is smaller and traders trade more frequently. This result is consistent with Figures 4 and 6, where price dampening is also more substantial for smaller h .

Our theoretical predictions are consistent with the stylized facts about cross-sectional patterns of time-series momentum. Lee and Swaminathan (2000) point out that momentum tends to be stronger for stocks with higher turnover. Cremers and Pareek (2014) find more substantial time-series momentum in stocks with more short-term trading. Moskowitz, Ooi, and Pedersen (2012) show that more liquid contracts in equity-index, currency, commodity, and bond futures markets exhibit greater momentum. Zhang (2006) and Verardo (2009) find that momentum returns are larger for stocks with higher analysts' disagreement. The predictions of our model are consistent with these stylized facts.

The right panel of Figure 7 plots the variance of holding-period return $\text{Var}[R_{k,k+f}]$ against

¹⁰In Figure 7, parameter values r , A , N , α_D , α_G , σ_D , and σ_G are given in Table 2 in Section 4. Parameters τ_H and τ_L are given in Table 2 and trading intervals are $h = 0.1, 0.2, 0.5$, and 1 in the left panel. Parameter $\tau = 2.2$ and the holding period $T = 1/12$ in the right panel.

C_G by varying the disagreement level τ_H/τ_L while holding τ fixed. It shows that return volatility is smaller when investors become more overconfident (τ_H/τ_L increases and thus C_G decreases).¹¹ This implies that traders' short-term speculative trading tends to dampen, rather than magnify, price fluctuations.

4 Model Calibration

We next calibrate model parameter values to fit well-known empirical return autocorrelations. We consider a sample of all common stocks listed on the NYSE and AMEX during the period January 1965 through December 2006 with at least two years of prior data.¹²

Let i subscript stocks and k subscript months. Let $R_{k-12,k,i}$ denote the annual return for the year prior to the end of month k , and $R_{k+f-12,k+f,i}$ denote the annual return over the year ending at the end of month $k+f$.

Column A of Table 1 presents time-series average slope coefficients from monthly cross-sectional regressions of one-year returns for 1–5 years ahead ($f = 12, 24, 36, 48, 60$ months) on lagged one-year returns:

$$(\text{Model A}) \quad R_{k+f-12,k+f,i} = a_{f,12} + b_{f,12} R_{k-12,k,i} + \varepsilon_{k+f-12,k+f,i}.$$

The methodology follows Fama and MacBeth (1973). Standard errors are computed using the correction of Hansen and Hodrick (1980). Estimated slope coefficients are positive and significant for 12 months; negative and insignificant for 24, 36, 48 months; and negative and significant for 60 months. These results are similar to the estimates of Lee and Swaminathan (2000) (column 1 of Table VIII). They suggest time-series momentum in year 1 and reversal by year 5.

Column B of Table 1 presents time-series average slope coefficients from monthly Fama–MacBeth cross-sectional regressions of cumulative returns $R_{k,k+f,i}$ for 1–5 years ($f = 12, 24, 36, 48, 60$ months) on lagged one-year returns $R_{k-12,k,i}$.

$$(\text{Model B}) \quad R_{k,k+f,i} = a_{f,12} + b_{f,12} R_{k-12,k,i} + \varepsilon_{k,k+f,i}.$$

¹¹It can be shown analytically that the instantaneous return variance increases in C_G from Equation (A-39) in the Appendix.

¹²We conduct our analysis using the sample period of 1965–2006 to exclude the financial crisis period. We also exclude companies incorporated outside the United States, Americus Trust Components (Primes and Scores), closed-end funds, and real estate investment trusts.

Table 1: Regression Tests of Return Momentum and Reversal. This table reports time-series averages of slope coefficients estimated from monthly Fama–MacBeth cross-sectional regressions of models A and B over 444 months from January 1965 to December 2001 ($k = 1, \dots, 444$). Model A is a regression of one-year returns from month $k + f - 12$ to $k + f$ on one-year lagged returns from month $k - 12$ to k . Model B is a regression of cumulative returns over f months from month k to $k + f$ on lagged one-year returns from month $k - 12$ to k . The coefficients are time-series means. The t -statistics in parentheses use the Hansen–Hodrick correction.

f (months)	Time-Series Average Slope Coefficients $b_{f,12}$			
	Model A		Model B	
	Coeff	t -stat	Coeff	t -stat
12	0.0463	(2.9338)	0.0463	(2.9338)
24	-0.0302	(-1.6528)	0.0109	(0.3326)
36	-0.0190	(-1.0088)	-0.0110	(-0.2509)
48	-0.0265	(-1.5759)	-0.0549	(-1.1776)
60	-0.0366	(-2.1726)	-0.1076	(-1.5957)

The coefficients reveal similar short-run time-series momentum and long-run mean-reversion.

To generate both short-term momentum and long-run mean-reversion, the calibration assumes that traders may have incorrect beliefs about the decay rate of the signal ($\hat{\alpha}_G + \hat{\tau} \neq \alpha_G + \tau$), and they agree about the instantaneous volatility of the growth rate ($\sigma_G = \hat{\sigma}_G$). In this general case, the expected holding-period excess return from time t to $t + T$ is given as¹³

$$\begin{aligned} \hat{E}_t[R(t, t+T)] = & \beta_1(T) \left(P(t) - \frac{D(t)}{r + \alpha_D} \right) - \beta_0(T) \int_{u=-\infty}^t e^{-(\hat{\alpha}_G + \hat{\tau})(t-u)} dI_0(u) \\ & + \beta_2(T) \int_{u=-\infty}^t \left(P(u) - \frac{D(u)}{r + \alpha_D} \right) e^{-(\hat{\alpha}_G + \hat{\tau})(t-u)} du. \end{aligned} \quad (53)$$

Coefficients $\beta_1(T)$, $\beta_2(T)$, and $\beta_0(T)$ in equation (53) are defined as

$$\beta_1(T) := \left(\zeta_2(T) \frac{\hat{\tau}_I^{1/2}}{\tau_I^{1/2}} - \zeta_1(T) \right) \frac{(r + \alpha_D)(r + \alpha_G)}{C_G \sigma_G \Omega^{1/2}}, \quad \beta_0(T) := \zeta_2(T) \frac{\hat{\tau}_I^{1/2} \tau_0^{1/2} - \tau_I^{1/2} \hat{\tau}_0^{1/2}}{\tau_I^{1/2}},$$

¹³If $\hat{\alpha}_G + \hat{\tau} \neq \alpha_G + \tau$, then $\hat{H}_n(t) \neq H_n(t)$, and the relationship between the two sufficient statistics depends on the entire history of information flow (the history of publicly observable dividends and prices), as shown in equation (A-36) in the Appendix. Therefore, we assume a trading interval $h \rightarrow 0$ for the calibration.

Table 2: The table reports the given parameter values for r , A , N , α_D , σ_D , α_G , σ_G , τ_H , and τ_L . The bottom two lines report the estimated parameters $\hat{\alpha}_G$ and $\hat{\tau}$.

Parameter	Description	Value
r	Risk-free rate	0.01
A	Risk aversion	1
N	Numbers of traders	100
α_D	Mean-reversion rate of dividend	0.04
σ_D	Instantaneous volatility of dividend	0.5
α_G	Traders' mean-reversion rate of $G(t)$	0.2
σ_G	Instantaneous volatility of $G(t)$	0.1
τ_H	Precision of trader n 's signal	0.3
τ_L	Precision of others' signal	0.02
$\hat{\alpha}_G$	Empirically correct mean-reversion rate of $G(t)$	0.292
$\hat{\tau}$	Empirically correct total precision	6.034

$$\beta_2(T) := \zeta_2(T) \frac{\hat{\tau}_I^{1/2}}{\tau_I^{1/2}} \frac{(\alpha_G + \tau - \hat{\alpha}_G - \hat{\tau})(r + \alpha_D)(r + \alpha_G)}{C_G \sigma_G \Omega^{1/2}}, \quad (54)$$

where $\zeta_1(T) > 0$ and $\zeta_2(T) > 0$ are defined in equation (A-43) in the Appendix. The coefficient $\beta_1(T)$ might be positive or negative depending on parameter values. The coefficient $\beta_0(T)$ is positive if and only if $\hat{\tau}_I^{1/2} \tau_0^{1/2} > \tau_I^{1/2} \hat{\tau}_0^{1/2}$. In addition, the coefficient $\beta_2(T)$ is negative if and only if $\alpha_G + \tau < \hat{\alpha}_G + \hat{\tau}$. A positive $\beta_1(T)$ or a positive $\beta_2(T)$ is related to time-series return momentum. A negative $\beta_1(T)$ or a negative $\beta_2(T)$ or a positive $\beta_0(T)$ implies mean reversion in returns. Therefore, different from the case where $\hat{\alpha}_G + \hat{\tau} = \alpha_G + \tau$ and thus $\beta_2(T) = 0$ (equation (52) in Theorem 5), in the general case, the expected holding-period excess return also depends on the past deviations of prices from the unconditional valuation ($\beta_2(T) \neq 0$); the importance of each past component decays exponentially at rate $\hat{\alpha}_G + \hat{\tau}$. As a result, the general case can generate rich patterns of return dynamics.¹⁴

To calibrate the structural model, we find model parameters that match theoretical regression coefficients to corresponding empirically estimated values in column B of Table 1.

Proposition 6 implies that the risk aversion parameter A scales trading volume but has

¹⁴Our structural model essentially imposes testable non-linear micro-founded economic restrictions on VAR models of expected returns such as Goyal and Welch (2003), Ang and Bekaert (2007), Cochrane (2008), Van Binsbergen and Koijen (2010), and Rytchkov (2012); these restrictions are sufficiently flexible to be consistent with the rich patterns of short-term momentum and long-term mean-reversion.

no effect on prices; we therefore assume $A = 1$. We assume $r = 0.01$ and $\alpha_D = 0.04$, implying $1/(r + \alpha_D) = 20$.¹⁵ We also assume the dividend volatility $\sigma_D = 0.5$ and the instantaneous volatility of the growth rate $\sigma_G = \hat{\sigma}_G = 0.1$.¹⁶ There remain six parameters: N , τ_H , τ_L , α_G , $\hat{\alpha}_G$ and $\hat{\tau}$. As intuition suggests, the magnitude and horizon of return momentum and reversal are determined by the level of disagreement τ_H/τ_L and the difference between empirically correct beliefs and traders' beliefs about the information decay rate ($\hat{\alpha}_G - \alpha_G$ and $\hat{\tau} - \tau$).

To further reduce the number of free parameters, we assume $N = 100$, $\tau_H = 0.3$, $\tau_L = 0.02$, and $\alpha_G = 0.2$; traders are overconfident and they actively engage in short-term trading against others in a Keynesian beauty contest. The mean-reversion rate $\alpha_G = 0.2$ implies that the growth rate $G(t)$ mean reverts about $\ln(2)/0.2 \approx 3.5$ years.¹⁷ We calibrate the two remaining parameters by matching empirically estimated values in column B of Table 1 to theoretically predicted regression coefficients. The calibrated parameter values are $\hat{\alpha}_G = 0.292$ and $\hat{\tau} = 6.034$. The calibrated parameter values imply that the coefficient $\beta_1(T) > 0$, $\beta_2(T) < 0$, and $\beta_0(T) > 0$ in equation (53). Consistent with our previous discussion, a positive $\beta_1(T)$ is related to time-series return momentum and a negative $\beta_2(T)$ and a positive $\beta_0(T)$ imply mean reversion in returns. The implied values of the two price dampening factors are $C_J = 0.96$ and $C_G = 0.54$. Time-series momentum arises from both effects, but the dynamic dampening effect of C_G is much larger than the static dampening effect of C_J . Time-series mean reversion arises because traders think that the signals decay at a slower rate than the empirically correct parameter ($\alpha_G + \tau < \hat{\alpha}_G + \hat{\tau}$). Table 2 summarizes the exogenously given and calibrated parameter values.

Figure 8 shows that the regression coefficients in our calibrated model closely match the empirical numbers. The ten dots correspond to ten empirical slope estimates from regression model B for $f = 6, 12, 18, 24, 30, 36, 42, 48, 54$, and 60 months. The solid curve depicts the theoretical regression coefficients for the calibrated model for $f = 6, 12, 18, 24, 30, 36, 42, 48, 54$, and 60 months. These theoretical predictions closely track all ten empirical estimates.

In online Appendix B.3, we discuss how our calibration results are affected by parameter

¹⁵If the growth rate is zero, then $P(t) = D(t)/(r + \alpha_D)$. We assume that a firm pays out all earnings as dividends, then the P/E ratio is approximately $1/(r + \alpha_D)$. If $r = 0.01$ and $\alpha_D = 0.04$, then the implied P/E ratio is 20. The median of Shiller P/E ratio for the S&P 500 is 15.76 and the current Shiller P/E ratio is 30.2. As illustrated in online Appendix B.3, the calibration results are not sensitive to the particular choices of α_D .

¹⁶Rountree, Weston, and Allayannis (2008) document that the average (median) standard deviation of quarterly earnings per share is 0.72 (0.19). Koren and Tenreyro (2007) document that the standard deviation of growth rates varies between 0.02 and 0.15.

¹⁷In online Appendix B.3, we also calibrate the model assuming $\alpha_G = 0.1$, implying that the growth rate mean reverts about $\ln(2)/0.1 \approx 7$ years.

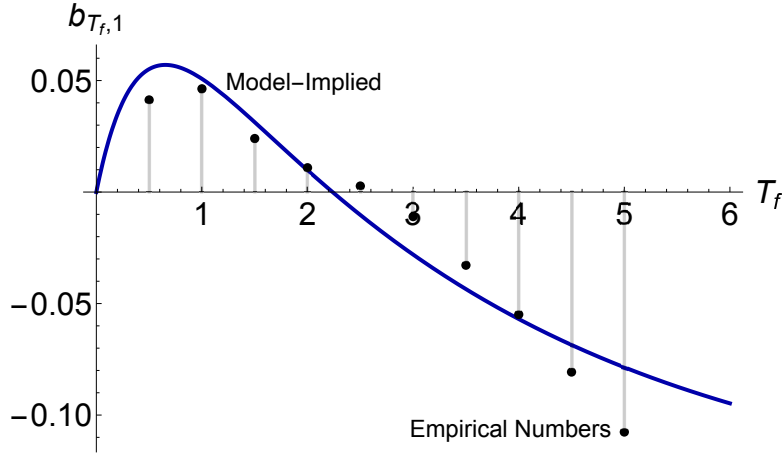


Figure 8: The figure shows theoretical regression coefficients (solid curve) and empirically estimated coefficients (dots) for different holding periods $T_f = f/12$ years (or f months).

values. The calibrated parameter values for $\hat{\alpha}_G$ and $\hat{\tau}$ are affected by different values of disagreement τ_H/τ_L , α_G , and σ_G as expected. When exogenous values of τ_H , τ_L , N , α_G , and σ_G are changed, we show that the model can still generate empirically realistic patterns with different calibrated parameter values for $\hat{\alpha}_G$ and $\hat{\tau}$.

5 Conclusion

When traders disagree about future valuations, a competitive market aggregates this disagreement in a manner that induces time-series momentum in returns. We clarify the underlying mechanism by first using a two-period model and then developing a dynamic trading model in which traders have heterogeneous beliefs about how to interpret continuous flows of privately observed information and trade at discrete trading rounds.

Even though all traders apply Bayes law consistently, they regularly spot profitable opportunities and think that they can profit at the expense of others. Even though prices fully reflect the average of all signals at each point in time, the prices are dampened relative to the average of traders' buy-and-hold valuations. Price dampening occurs as a result of traders taking advantage of perceived short-term speculative trading opportunities rather than trading solely on long-term valuations of fundamentals. Contrary to the Keynesian intuition, this short-term speculative trading, when internalized by the market, leads to price dampening, not excessive volatility. When trading rounds occur less frequently, opportunities to engage in short-term

speculation are reduced, and there is less dampening. We calibrate model parameter values to demonstrate that our model can generate quantitatively realistic empirical patterns of return dynamics.

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A Proofs

A.1 Proof of Theorem 1

If each trader holds conditional mean-variance optimal portfolio at each period, then

$$\check{S}_{n1} = \frac{E_1^n[\check{P}_2] - \check{P}_1}{A \text{Var}_1^n[\check{P}_2]}, \quad \check{S}_{n2} = \frac{E_2^n[\nu] - \check{P}_2}{A \text{Var}_2^n[\nu]}.$$

Market clearing condition $\sum_{n=1}^N \check{S}_{nt} = 0$ ($t = 1, 2$) implies that the equilibrium prices

$$\check{P}_1 = \frac{1}{N} \sum_{n=1}^N E_1^n[\check{P}_2], \quad \check{P}_2 = \frac{1}{N} \sum_{n=1}^N E_2^n[\nu].$$

Simple calculation shows that $E_1^n[\check{P}_2]$ is

$$E_1^n[\check{P}_2] = E_1^n\left[\frac{1}{N} \sum_{n=1}^N E_2^n[\nu]\right] = \frac{\tau_1}{\tau_2} \frac{1}{N} \sum_{n=1}^N E_1^n[\nu] + \frac{\tau_\nu}{N\tau_2} \left(\tau_H^{1/2} + (N-1)\tau_L^{1/2}\right)^2 E_1^n[\nu].$$

It follows that

$$\check{P}_1 = \frac{1}{N} \sum_{n=1}^N E_1^n[\check{P}_2] = \left(1 - \frac{\tau_\nu}{\tau_2} \left(1 - \frac{1}{N}\right) \left(\tau_H^{1/2} - \tau_L^{1/2}\right)^2\right) \frac{1}{N} \sum_{n=1}^N E_1^n[\nu] = \check{C}_g \frac{1}{N} \sum_{n=1}^N E_1^n[\nu],$$

where the coefficient $\check{C}_g$ satisfies $0 < \check{C}_g < 1$ and is given as in equation (5).

A.2 Proof of Theorem 2

At $t = 1, 2$, trader n chooses quantity S_{nt} to maximize the expected utility of terminal wealth $E_t^n[-e^{-AW_{n3}}]$. At $t = 2$, the optimal inventory $S_{n2} = (E_2^n[\nu] - P_2)/(A \text{Var}_2^n[\nu])$. The market-clearing condition $\sum_{n=1}^N S_{n2} = 0$ yields the market clearing price at $t = 2$:

$$P_2 = \frac{1}{N} \sum_{n=1}^N E_2^n[\nu] = \frac{\tau_v^{1/2}}{\tau_2} \left(\tau_H^{1/2} + (N-1) \tau_L^{1/2} \right) \frac{1}{N} \sum_{n=1}^N (i_{n1} + i_{n2}).$$

At $t = 1$, trader n chooses optimal demand S_{n1} to maximize

$$E_1^n \left[-\exp \left(-A \left(W_{n1} + (P_2 - P_1) S_{n1} + \frac{(E_2^n[\nu] - P_2)^2}{2A \text{Var}_2^n[\nu]} \right) \right) \right], \quad (\text{A-1})$$

where $P_2 - P_1$ and $E_2^n[\nu] - P_2$ can be expressed as

$$\begin{aligned} P_2 - P_1 &= (P_2 - E_1^n[P_2]) + (E_1^n[P_2] - P_1), \\ E_2^n[\nu] - P_2 &= (E_2^n[\nu] - E_1^n[\nu]) - (P_2 - E_1^n[P_2]) + (E_1^n[\nu] - E_1^n[P_2]). \end{aligned}$$

Defining

$$y := [P_2 - E_1^n[P_2], E_2^n[\nu] - E_1^n[\nu]]',$$

then the solution for the problem (A-1) can be obtained using

$$E_1^n \left[e^{-\alpha \left(\bar{A} + B'y + \frac{1}{2} y' C y \right)} \right] = \frac{1}{\sqrt{|I + \alpha C \Sigma|}} e^{-\alpha \left(\bar{A} - \frac{1}{2} \alpha B' \Sigma (I + \alpha C \Sigma)^{-1} B \right)},$$

where y is a 2×1 vector of zero-mean normally distributed random variables with covariance matrix Σ ,

$$\Sigma := \begin{bmatrix} \text{Var}_1^n[P_2] & \text{Cov}_1^n[P_2, E_2^n[\nu]] \\ \text{Cov}_1^n[P_2, E_2^n[\nu]] & \text{Var}_1^n[E_2^n[\nu]] \end{bmatrix},$$

$\bar{A}$ is a scalar, B is a 2×1 vector, C is a 2×2 symmetric matrix, and I is the 2×2 identity matrix. Define α , $\bar{A}$, B , and C as

$$\begin{aligned} \alpha &:= A, \quad \bar{A} := W_{n1} + (E_1^n[P_2] - P_1) S_{n1} + \frac{(E_1^n[\nu] - E_1^n[P_2])^2}{2A \text{Var}_2^n[\nu]}, \\ B &:= \psi_B x_{n1} + \frac{E_1^n[\nu] - E_1^n[P_2]}{A \text{Var}_2^n[\nu]} \phi_B, \quad C := \frac{1}{A \text{Var}_2^n[\nu]} \times \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \end{aligned}$$

where $\psi_B := [1, 0]'$ and $\phi_B := [-1, 1]'$. The first-order condition for S_{n1} implies

$$S_{n1} = \frac{E_1^n[P_2] - P_1 - \frac{E_1^n[v] - E_1^n[P_2]}{\text{Var}_2^n[v]} \psi_B' \Sigma (I + AC\Sigma)^{-1} \phi_B}{A \psi_B' \Sigma (I + AC\Sigma)^{-1} \psi_B}. \quad (\text{A-2})$$

It can be shown that

$$\begin{aligned} \frac{\psi_B' \Sigma (I + AC\Sigma)^{-1} \phi_B}{\text{Var}_2^n[v]} &= \frac{\text{Cov}_1^n[P_2, E_2^n[v] - P_2]}{\text{Var}_1^n[E_2^n[v] - P_2] + \text{Var}_2^n[v]}, \\ \psi_B' \Sigma (I + AC\Sigma)^{-1} \psi_B &= \frac{\text{Var}_1^n[P_2] (\text{Var}_1^n[E_2^n[v]] + \text{Var}_2^n[v]) - (\text{Cov}_1^n[P_2, E_2^n[v]])^2}{\text{Var}_1^n[E_2^n[v] - P_2] + \text{Var}_2^n[v]}. \end{aligned} \quad (\text{A-3})$$

Substituting equation (A-3) into equation (A-2) yields trader n 's optimal inventory as given in equation (11) in Theorem 2. The market-clearing condition $\sum_{n=1}^N S_{n1} = 0$ yields

$$P_1 = \frac{1}{N} \sum_{n=1}^N E_1^n[P_2] + \frac{\text{Cov}_1^n[P_2, E_2^n[v] - P_2]}{\text{Var}_1^n[E_2^n[v] - P_2] + \text{Var}_2^n[v]} \left(\frac{1}{N} \sum_{n=1}^N E_1^n[P_2] - \frac{1}{N} \sum_{n=1}^N E_1^n[v] \right).$$

It can be shown that P_1 can be written as $C_g \times \frac{1}{N} \sum_{n=1}^N E_1^n[v]$ where the coefficient C_g is

$$\begin{aligned} C_g &= \frac{\tau_1}{\tau_2} + \frac{\tau_v}{\tau_2} \frac{1}{N} \left(\tau_H^{1/2} + (N-1) \tau_L^{1/2} \right)^2 \frac{N \tau_1 \left(N \tau_2 + (N-1) \tau_v \left(\tau_H^{1/2} - \tau_L^{1/2} \right)^2 \right)}{N \tau_1 \left(N \tau_2 + (N-1) \tau_v \left(\tau_H^{1/2} - \tau_L^{1/2} \right)^2 \right) + (N-1)^2 \tau_v^2 \left(\tau_H^{1/2} - \tau_L^{1/2} \right)^4} \\ &\leq \frac{\tau_1}{\tau_2} + \frac{\tau_v}{\tau_2} \frac{1}{N} \left(\tau_H^{1/2} + (N-1) \tau_L^{1/2} \right)^2 \leq \frac{\tau_1}{\tau_2} + \frac{\tau_v}{\tau_2} (\tau_H + (N-1) \tau_L) = 1. \end{aligned} \quad (\text{A-4})$$

Equation (A-4) implies that $C_g < 1$ if $\tau_H > \tau_L$ and $C_g = 1$ if $\tau_H = \tau_L$. Thus, P_1 is a weighted average of traders' valuations, with weights summing to less than one.

A.3 Proof of Proposition 1

From equation (15), we have

$$C_g = \check{C}_g - \frac{\tau_v}{\tau_2} \left(1 - \frac{1}{N} \right) \left(\tau_H^{1/2} - \tau_L^{1/2} \right)^2 \frac{\text{Cov}_1^n[P_2, E_2^n[v] - P_2]}{\text{Var}_1^n[E_2^n[v] - P_2] + \text{Var}_2^n[v]},$$

where $\text{Cov}_1^n[P_2, E_2^n[v] - P_2] > 0$ from equation (13). This implies $C_g < \check{C}_g$. From equation (A-4), it is straightforward to see that C_g satisfies $0 < C_g < 1$ for $\tau_H \neq \tau_L$. Using direct calculation, we show analytically that $dC_g/d\tau_H < 0$ and $d\check{C}_g/d\tau_H < 0$ holding the total precision $\tau_H + (N-1)\tau_L$ fixed. It implies that both C_g and $\check{C}_g$ decrease in τ_H/τ_L holding the total precision fixed.

A.4 Proof of Proposition 2

Under traders' beliefs, the return auto-covariances are

$$\begin{aligned} \text{Cov}^n[\nu - P_2, P_2 - P_1] &= \text{Cov}^n\left[\nu - \frac{1}{N} \sum_{n=1}^N E_2^n[\nu], \frac{1}{N} \sum_{n=1}^N E_2^n[\nu] - C_g \times \frac{1}{N} \sum_{n=1}^N E_1^n[\nu]\right] \\ &= \frac{2\tau_\nu}{N^2\tau_2} \left(\frac{2}{\tau_2} - \frac{C_g}{\tau_1}\right) \left(\tau_H^{1/2} + (N-1)\tau_L^{1/2}\right)^2 (N-1) \left(\tau_H^{1/2} - \tau_L^{1/2}\right)^2, \\ \text{Cov}^n[P_2 - P_1, P_1 - P_0] &= \frac{\left(\tau_H^{1/2} + (N-1)\tau_L^{1/2}\right)^2}{N} \frac{\tau_\nu C_g}{\tau_1} \left(\left(\frac{2}{\tau_2} - \frac{C_g}{\tau_1}\right) \frac{\left(\tau_H^{1/2} + (N-1)\tau_L^{1/2}\right)^2}{N} + \left(\frac{1}{\tau_2} - \frac{C_g}{\tau_1}\right)\right). \end{aligned} \quad (\text{A-5})$$

We show analytically that both $\text{Cov}^n[\nu - P_2, P_2 - P_1] > 0$ and $\text{Cov}^n[P_2 - P_1, P_1 - P_0] > 0$.

Assume traders are correct on average. Under the empirically correct parameters that all private signals have the same precision, prices have the same expressions as the ones under traders' beliefs (equation (16)). Under the empirically correct parameters that all private signals have the same precision, however, traders' private signals are

$$\hat{i}_{nt} = \left(\frac{\tau_H + (N-1)\tau_L}{N}\right)^{1/2} (\tau_\nu^{1/2} \nu) + e_{nt}, \quad \hat{i}_{mt} = \left(\frac{\tau_H + (N-1)\tau_L}{N}\right)^{1/2} (\tau_\nu^{1/2} \nu) + e_{mt}, \quad \text{for all } m \neq n.$$

Using the empirically correct parameters, the return auto-covariance is

$$\begin{aligned} \text{Cov}[P_2 - P_1, P_1 - P_0] &= \frac{\tau_\nu C_g}{\tau_1} \left(\left(\frac{2}{\tau_2} - \frac{C_g}{\tau_1}\right) \frac{\left(\tau_H^{1/2} + (N-1)\tau_L^{1/2}\right)^2}{N} \frac{(\tau_H + (N-1)\tau_L)}{N} + \left(\frac{1}{\tau_2} - \frac{C_g}{\tau_1}\right) \frac{\left(\tau_H^{1/2} + (N-1)\tau_L^{1/2}\right)^2}{N}\right), \\ \text{Cov}[\nu - P_2, P_2 - P_1] &= \left(\frac{2}{\tau_2} - \frac{C_g}{\tau_1}\right) \frac{\tau_\nu}{N\tau_2} \left(N^{1/2} (\tau_H + (N-1)\tau_L)^{1/2} - (\tau_H^{1/2} + (N-1)\tau_L^{1/2})\right) \\ &\quad \times (\tau_H^{1/2} + (N-1)\tau_L^{1/2}) (1 + 2(\tau_H + (N-1)\tau_L)). \end{aligned} \quad (\text{A-6})$$

Simple calculation shows that $\text{Cov}[\nu - P_2, P_2 - P_1] > 0$ and $\text{Cov}[P_2 - P_1, P_1 - P_0] > 0$.

A.5 Proof of Proposition 3

If traders agree about the conditional distribution of ν given signals, then for $t = 1, 2$, $E_t^n[\nu] = E_t^m[\nu]$. Thus, $E_1^n[E_2^m[\nu]] = E_1^n[E_2^n[\nu]] = E_1^n[\nu]$ where the last equation follows from iteration of trader n 's expectations. It follows that $E_1^n\left[\frac{1}{N} \sum_{m=1}^N E_2^m[\nu]\right] = E_1^n[\nu] = \frac{1}{N} \sum_{m=1}^N E_1^m[\nu]$, and thus $P_1 = E_1^n[P_2]$.

If traders agree about the joint distribution of signals, then $E_1^n[E_2^m[\nu]] = E_1^m[E_2^m[\nu]]$ because $E_2^m[\nu]$ is some known function of observed signals that trader m uses in his filtering. Then $E_1^n[E_2^m[\nu]] = E_1^m[E_2^m[\nu]] = E_1^m[\nu]$, where the last equation follows from iteration of trader m 's expectations. It follows that $E_1^n\left[\frac{1}{N} \sum_{m=1}^N E_2^m[\nu]\right] = \frac{1}{N} \sum_{m=1}^N E_1^m[\nu]$, and thus $P_1 = E_1^n[P_2]$.

A.6 Filtering Formulas for the Dynamic Model

In this subsection, we derive Stratonovich–Kalman–Bucy filtering formulas for Ω and $G_{n,k}$.

Lemma 1. *Define the total precision of information flows as $\tau := \tau_0 + \tau_H + (N-1)\tau_L$. Then, the scaled steady state error variance Ω and trader n 's estimate $G_{n,k}$ of the growth rate at time t are*

$$\Omega^{-1} := \text{Var}^{-1} \left[\frac{G^*(t) - G_n(t)}{\sigma_G} \right] = 2\alpha_G + \tau, \quad (\text{A-7})$$

$$G_n(t) := E_t^n [G^*(t)] = \sigma_G \Omega^{1/2} \left(\tau_0^{1/2} H_0(t) + \tau_H^{1/2} H_n(t) + (N-1)\tau_L^{1/2} H_{-n}(t) \right). \quad (\text{A-8})$$

Proof: The stochastic process $dG^*(t)$, $dI_n(t)$, $dI_{-n}(t)$, and $dI_0(t)$ can be rewritten as

$$\begin{aligned} dG^*(t) &:= -\alpha_G G_n(t) dt - \alpha_G (G^*(t) - G_n(t)) dt + \sigma_G dB_G(t), \\ dI_n(t) &:= \tau_H^{1/2} \frac{G_n(t)}{\sigma_G \Omega^{1/2}} dt + \tau_H^{1/2} \frac{G^*(t) - G_n(t)}{\sigma_G \Omega^{1/2}} dt + dB_n(t), \\ dI_0(t) &:= \tau_0^{1/2} \frac{G_n(t)}{\sigma_G \Omega^{1/2}} dt + \tau_0^{1/2} \frac{G^*(t) - G_n(t)}{\sigma_G \Omega^{1/2}} dt + dB_0(t), \\ dI_{-n}(t) &:= \tau_L^{1/2} \frac{G_n(t)}{\sigma_G \Omega^{1/2}} dt + \tau_L^{1/2} \frac{G^*(t) - G_n(t)}{\sigma_G \Omega^{1/2}} dt + \frac{1}{N-1} \sum_{m=1, m \neq n}^N dB_m(t). \end{aligned} \quad (\text{A-9})$$

Since trader n 's forecast of the error $G^*(t) - G_n(t)$ is zero given his information set, the last two terms in the information processes are independently distributed Brownian motions from the perspective of trader n . Let $I(u)$ denote trader n 's information set. Trader n 's estimate at time $t + dt$, $G_n(t + dt) := E^n [G^*(t + dt) | I(u)]_{u=-\infty}^{t+dt}$, can be decomposed into two terms:

$$G_n(t + dt) = E^n [G^*(t) | I(u)]_{u=-\infty}^{t+dt} + E^n [dG^*(t) | I(u)]_{u=-\infty}^{t+dt}. \quad (\text{A-10})$$

The first term reflects expectation about the past growth rate $G^*(t)$ at time t given new information $dI_0(t)$, $dI_n(t)$, and $dI_{-n}(t)$. The second term reflects expectation about the change in growth rate $dG^*(t) = G^*(t + dt) - G^*(t)$ at time t given new information $dI_0(t)$, $dI_n(t)$, and $dI_{-n}(t)$. The first term $E^n [G^*(t) | I(u)]_{u=-\infty}^{t+dt}$ can be calculated as follows:

$$\begin{aligned}
\mathbb{E}^n \left[G^*(t) \middle| I(u) \Big|_{u=-\infty}^{t+\mathrm{d}t} \right] &= \mathbb{E}^n [G_n(t) + (G^*(t) - G_n(t)) \mid G_n(t), \mathrm{d}I_0(t), \mathrm{d}I_n(t), \mathrm{d}I_{-n}(t)] \\
&= G_n(t) + \mathbb{E}^n \left[G^*(t) - G_n(t) \middle| \mathrm{d}I_0(t) - \frac{\tau_0^{1/2} G_n(t)}{\sigma_G \Omega^{1/2}} \mathrm{d}t, \mathrm{d}I_n(t) - \frac{\tau_H^{1/2} G_n(t)}{\sigma_G \Omega^{1/2}} \mathrm{d}t, \mathrm{d}I_{-n}(t) - \frac{\tau_L^{1/2} G_n(t)}{\sigma_G \Omega^{1/2}} \mathrm{d}t \right] \\
&= G_n(t) + \sigma_G \Omega^{1/2} \tau_0^{1/2} \left(\mathrm{d}I_0(t) - \frac{\tau_0^{1/2} G_n(t)}{\sigma_G \Omega^{1/2}} \mathrm{d}t \right) + \sigma_G \Omega^{1/2} \tau_H^{1/2} \left(\mathrm{d}I_n(t) - \frac{\tau_H^{1/2} G_n(t)}{\sigma_G \Omega^{1/2}} \mathrm{d}t \right) \\
&\quad + \sigma_G \Omega^{1/2} (N-1) \tau_L^{1/2} \left(\mathrm{d}I_{-n}(t) - \frac{\tau_L^{1/2} G_n(t)}{\sigma_G \Omega^{1/2}} \mathrm{d}t \right).
\end{aligned} \tag{A-11}$$

The last equation is obtained using the projection theorem for normal variables and equation (A-9). The second term $\mathbb{E}^n \left[\mathrm{d}G^*(t) \middle| I(u) \Big|_{u=-\infty}^{t+\mathrm{d}t} \right]$ can be calculated as follows:

$$\begin{aligned}
\mathbb{E}^n \left[\mathrm{d}G^*(t) \middle| I(u) \Big|_{u=-\infty}^{t+\mathrm{d}t} \right] &= \mathbb{E}^n [-\alpha_G G_n(t) \mathrm{d}t + \mathrm{d}G^*(t) + \alpha_G G_n(t) \mathrm{d}t \mid G_n(t), \mathrm{d}I_0(t), \mathrm{d}I_n(t), \mathrm{d}I_{-n}(t)] \\
&= -\alpha_G G_n(t) \mathrm{d}t - \alpha_G \mathbb{E}^n [(G^*(t) - G_n(t)) \mathrm{d}t \mid G_n(t), \mathrm{d}I_0(t), \mathrm{d}I_n(t), \mathrm{d}I_{-n}(t)].
\end{aligned} \tag{A-12}$$

The second term in equation (A-12) is as calculated above in equation (A-11). It can be ignored since it is of order smaller than $\mathrm{d}t$. Plugging equations (A-11) and (A-12) into equation (A-10), we find that the estimate $G_n(t)$ is defined by the Itô differential equation

$$\mathrm{d}G_n(t) = -\alpha_G G_n(t) \mathrm{d}t + \sum_{n=0}^N \sigma_G \Omega^{1/2} \tau_n^{1/2} \left(\mathrm{d}I_n(t) - G_n(t) \frac{\tau_n^{1/2}}{\sigma_G \Omega^{1/2}} \mathrm{d}t \right).$$

Rearranging terms yields

$$\mathrm{d}G_n(t) = -(\alpha_G + \tau) G_n(t) \mathrm{d}t + \sigma_G \Omega^{1/2} \left(\tau_0^{1/2} \mathrm{d}I_0 + \tau_H^{1/2} \mathrm{d}I_n + (N-1) \tau_L^{1/2} \mathrm{d}I_{-n} \right). \tag{A-13}$$

From equation (A-13), we obtain the solution for the estimate $G_n(t)$:

$$G_n(t) = \sigma_G \Omega^{1/2} \int_{u=-\infty}^t e^{-(\alpha_G + \tau)(t-u)} \left(\tau_0^{1/2} \mathrm{d}I_0(u) + \tau_H^{1/2} \mathrm{d}I_n(u) + (N-1) \tau_L^{1/2} \mathrm{d}I_{-n}(u) \right) \mathrm{d}u.$$

Define the sufficient statistics or “signals” $H_n(t)$ by

$$H_n(t) := \int_{u=-\infty}^t e^{-(\alpha_G + \tau)(t-u)} \mathrm{d}I_n(u), \quad n = 0, 1, \dots, N.$$

The estimate $G_n(t)$ can be conveniently written as the weighted sum of these sufficient statis-

tics as in equation (A-8). The mean-square filtering error of the estimate $G_n(t)$, denoted $\sigma_G^2 \Omega(t)$, is defined by the Riccati differential equation

$$\sigma_G^2 \frac{d\Omega(t)}{dt} = -2\alpha_G \sigma_G^2 \Omega(t) + \sigma_G^2 - \sigma_G^4 \Omega(t)^2 \sum_{n=0}^N \left(\frac{\tau_n^{1/2}}{\sigma_G \Omega(t)^{1/2}} \right)^2.$$

Let Ω denote the steady state of the function $\Omega(t)$. Using the steady-state assumption $d\Omega(t)/dt = 0$, solve this equation for the steady state value $\Omega = \Omega(t)$ to obtain equation (A-7). Equation (A-8) implies that, in discrete notation, trader n 's estimate of the growth rate at time $t = kh$ is

$$G_{n,k} := E_{kh}^n [G^*(kh)] = \sigma_G \Omega^{1/2} \left(\tau_0^{1/2} H_{0,k} + \tau_H^{1/2} H_{n,k} + (N-1) \tau_L^{1/2} H_{-n,k} \right). \quad \square$$

A.7 Proof of Theorem 3 and Theorem 4

We first derive the dynamics of key state variables and then solve for the equilibrium.

A.7.1 The Dynamics of Key State Variables

We next derive conditional expectations of key state variables and their variance-covariance matrix. Define $N+1$ processes $dB_0^n(t)$, $dB_n^n(t)$, and $dB_m^n(t)$, $m = 1, \dots, N$, $m \neq n$, by

$$\begin{aligned} dB_0^n(t) &= \tau_0^{1/2} \frac{G^*(t) - G_n(t)}{\sigma_G \Omega^{1/2}} dt + dB_D(t), \quad dB_n^n(t) = \tau_H^{1/2} \frac{G^*(t) - G_n(t)}{\sigma_G \Omega^{1/2}} dt + dB_n(t), \\ dB_m^n(t) &= \tau_L^{1/2} \frac{G^*(t) - G_n(t)}{\sigma_G \Omega^{1/2}} dt + dB_m(t). \end{aligned}$$

The superscript n indicates conditioning on beliefs of trader n . Since trader n 's forecast of the error $G^*(t) - G_n(t)$ is zero given his information set, these $N+1$ processes are independently distributed Brownian motions from the perspective of trader n . In terms of these Brownian motions, trader n believes that sufficient statistics $H_0(t)$, $H_n(t)$ and $H_{-n}(t)$ change as follows:

$$\begin{aligned} dH_0(t) &= -(\alpha_G + \tau) H_0(t) dt + \tau_0^{1/2} \frac{G_n(t)}{\sigma_G \Omega^{1/2}} dt + dB_0^n(t), \\ dH_n(t) &= -(\alpha_G + \tau) H_n(t) dt + \tau_H^{1/2} \frac{G_n(t)}{\sigma_G \Omega^{1/2}} dt + dB_n^n(t), \\ dH_{-n}(t) &= -(\alpha_G + \tau) H_{-n}(t) dt + \tau_L^{1/2} \frac{G_n(t)}{\sigma_G \Omega^{1/2}} dt + \frac{1}{N-1} \sum_{m=1, m \neq n}^N dB_m^n(t). \end{aligned} \tag{A-14}$$

Combine the three sufficient statistics into the two sufficient statistics

$$H_n^c(t) := H_n(t) + \hat{a} H_0(t), \quad H_{-n}^c(t) := H_{-n}(t) + \hat{a} H_0(t), \quad \text{with} \quad \hat{a} := \frac{\tau_0^{1/2}}{\tau_H^{1/2} + (N-1)\tau_L^{1/2}}. \quad (\text{A-15})$$

Define $y(t) = [D(t), H_n^c(t), H_{-n}^c(t)]'$ as a continuous 3-vector stochastic process of all state variables. Using equation (A-14), write this stochastic vector in matrix form as

$$dy(t) = K y(t) dt + C_z dZ(t),$$

where K is a 3×3 matrix and C_z is a 3×3 matrix given by

$$K = \begin{bmatrix} -\alpha_D & \sigma_G \Omega^{1/2} \tau_H^{1/2} & \sigma_G \Omega^{1/2} (N-1) \tau_L^{1/2} \\ 0 & -\alpha_G - \tau + \tau_H^{1/2} (\tau_H^{1/2} + \hat{a} \tau_0^{1/2}) & (N-1) \tau_L^{1/2} (\tau_H^{1/2} + \hat{a} \tau_0^{1/2}) \\ 0 & \tau_H^{1/2} (\tau_L^{1/2} + \hat{a} \tau_0^{1/2}) & -\alpha_G - \tau + (N-1) \tau_L^{1/2} (\tau_L^{1/2} + \hat{a} \tau_0^{1/2}) \end{bmatrix}, \quad C_z = \begin{bmatrix} \sigma_D & 0 & 0 \\ \hat{a} & 1 & 0 \\ \hat{a} & 0 & \frac{1}{\sqrt{N-1}} \end{bmatrix},$$

and $dZ(t) = [dB_0^n(t), dB_n^n(t), \frac{1}{\sqrt{N-1}} \sum_{m=1, m \neq n}^N dB_m^n(t)]'$ is a 3-dimensional Brownian motion.

We can also represent the process $y_{k+1} = [D_{k+1}, H_{n,k+1}^c, H_{-n,k+1}^c]'$ as an integral

$$y_{k+1} = e^{Kh} y_k + \int_{kh}^{(k+1)h} e^{K((k+1)h-t)} C_z dZ(t).$$

This yields the following equations for conditional expectations and variance of y_{k+1} :

$$E_{kh}^n[y_{k+1}] = e^{Kh} [D_k, H_{n,k}^c, H_{-n,k}^c]', \quad \text{Var}_{kh}[y_{k+1}] = \int_0^h e^{K(h-t)} C_z C_z' e^{K'(h-t)} dt. \quad (\text{A-16})$$

Since trading is discrete, we also need to derive dynamics for variable $\check{D}_{k+1}$, defined in equation (30). In particular, we solve for $E_{kh}^n[\check{D}_{k+1}]$, $\text{Var}_{kh}[\check{D}_{k+1}]$, and $\text{Cov}[\check{D}_{k+1}, y_{k+1}]$. Define

$$\check{y}_{k+1} := e^{rh} \int_{kh}^{(k+1)h} e^{-r(t-kh)} y(t) dt.$$

It can be shown that

$$\check{y}_{k+1} = (K - rI)^{-1} \left((e^{(K-rI)h} - I) e^{rh} y_k + \int_{kh}^{(k+1)h} e^{K((k+1)h-t)} C_z dZ(t) - \int_{kh}^{(k+1)h} e^{-r(t-(k+1)h)} C_z dZ(t) \right).$$

The expectation $E_{kh}^n[\check{D}_{k+1}]$ is given by the first element in the 4×1 vector $e^{rh} (K - rI)^{-1} (e^{(K-rI)h} - I) y_k$ in the equation above. We can then derive

$$\begin{aligned}
\text{Cov}[y_{k+1}, \check{y}_{k+1}] &= \text{Var}_{kh}[y_{k+1}](K' - rI)^{-1} - (K + rI)^{-1}(e^{(K+rI)h} - I)C_z C_z'(K' - rI)^{-1}, \\
\text{Cov}[\check{y}_{k+1}, \check{y}_{k+1}] &= ((K - rI)^{-1}\text{Var}_{kh}[y_{k+1}] - (K - rI)^{-1}C_z C_z'(e^{(K+rI)h} - I)'(K' + rI)^{-1})(K' - rI)^{-1} \\
&\quad - \left((K^2 - r^2 I)^{-1}(e^{(K+rI)h} - I) + \frac{1 - e^{2rh}}{2r}(K - rI)^{-1} \right) C_z C_z'(K' - rI)^{-1}.
\end{aligned} \tag{A-17}$$

Then, $\text{Var}_k[\check{D}_{k+1}]$ is the (1,1) entry of the matrix $\text{Cov}[\check{y}_{k+1}, \check{y}_{k+1}]$, and $\text{Cov}[\check{D}_{k+1}, y_{k+1}]$ is given by the first column of the matrix $\text{Cov}[y_{k+1}, \check{y}_{k+1}]$.

A.7.2 Solution of the General Dynamic Model

We next solve for the equilibrium. We conjecture that the price in period k is a linear function of D_k and $\bar{G}_k$, of the form

$$P_k = \frac{D_k}{r + \alpha_D} + C_G \frac{\bar{G}_k}{(r + \alpha_D)(r + \alpha_G)}. \tag{A-18}$$

Trader n 's problem (27) can be rewritten in discrete-time form as (28), where $U_{n,j}$ is obtained by solving the maximization problem (31) subject to constraint (32). The first-order condition yields

$$c(jh + t) = -\frac{1}{A} \left((\rho - r)t + \ln \frac{\lambda}{A} \right), \tag{A-19}$$

where λ is the Lagrange multiplier. If $t = 0$ in equation (A-19), then $c(jh) = -\frac{1}{A} \ln \frac{\lambda}{A}$. Thus,

$$c(jh + t) = c(jh) + \frac{1}{A}(r - \rho)t. \tag{A-20}$$

Substituting (A-20) into constraint (32), we have

$$c(jh) = \frac{rh}{1 - e^{-rh}} c_{n,j} - \frac{r - \rho}{Ar(1 - e^{-rh})} (1 - (rh + 1)e^{-rh}). \tag{A-21}$$

From equations (A-20), (A-21), and (31), we get

$$U_{n,j} = -h \exp \left(-A \frac{rh}{1 - e^{-rh}} c_{n,j} \right) \phi(r, \rho, h),$$

where $\phi(r, \rho, h)$ is defined as

$$\phi(r, \rho, h) = \frac{1 - e^{-rh}}{rh} \exp \left(\frac{r - \rho}{r(1 - e^{-rh})} (1 - (1 + rh)e^{-rh}) \right) = 1 - \frac{1}{2}\rho h + O(h^2).$$

Trader n 's problem (28) is then equivalent to

$$\max_{\substack{[c_{n,j}], j=k, k+1, \dots, \infty \\ [S_{n,j}], j=k, k+1, \dots, \infty}} E_{kh}^n \sum_{j=k} -h e^{-\rho(j-k)h} \exp\left(-A \frac{rh}{1-e^{-rh}} c_{n,j}\right),$$

subject to the budget constraint (29).

We conjecture and verify that the value function has the specific quadratic exponential form

$$V_k(W_{n,k}, H_{n,k}^c, H_{-n,k}^c) = -\exp\left(\psi_0 + \psi_W W_{n,k} + \frac{1}{2}\psi_{nn}(H_{n,k}^c)^2 + \frac{1}{2}\psi_{xx}(H_{-n,k}^c)^2 + \psi_{nx}H_{n,k}^c H_{-n,k}^c\right). \quad (\text{A-22})$$

The five constants ψ_0 , ψ_W , ψ_{nn} , ψ_{xx} , and ψ_{nx} have values consistent with a steady-state equilibrium. The terms ψ_{nn} , ψ_{xx} , and ψ_{nx} capture the value of future trading opportunities based on current public and private information. The value of trading on innovations to future information is built into the constant term ψ_0 .

The Hamilton–Jacobi–Bellman (HJB) equation for the discrete problem is

$$V_k(W_{n,k}, H_{n,k}^c, H_{-n,k}^c) = \max_{\substack{c_{n,k} \\ S_{n,k}}} \left[-h \exp\left(-\frac{Arh}{1-e^{-rh}} c_{n,k}\right) + e^{-\rho h} E_{kh}^n V_{k+1}(W_{n,k+1}, H_{n,k+1}^c, H_{-n,k+1}^c) \right], \quad (\text{A-23})$$

where the dynamics of wealth satisfies (29) and the dynamics of $H_{n,k}^c$ and $H_{-n,k}^c$ can be obtained from equations (A-14) and (A-15). Define

$$x_{k+1} := [D_{k+1}, \check{D}_{k+1}, H_{n,k+1}^c, H_{-n,k+1}^c]' - [E_{kh}^n[D_{k+1}], E_{kh}^n[\check{D}_{k+1}], E_{kh}^n[H_{n,k+1}^c], E_{kh}^n[H_{-n,k+1}^c]]',$$

where $\check{D}_{k+1}$ is as defined in equation (30). Then $E_{kh}^n[V_{k+1}]$ can be obtained using

$$E_{kh}^n \left[e^{-\alpha(\bar{A} + B'x + \frac{1}{2}x'Cx)} \right] = \frac{1}{\sqrt{|I + \alpha C \Sigma|}} e^{-\alpha(\bar{A} - \frac{1}{2}\alpha B' \Sigma (I + \alpha C \Sigma)^{-1} B)},$$

where x is an $n \times 1$ normal vector with mean zero and covariance matrix Σ , $\bar{A}$ is a scalar, B is an $n \times 1$ vector, C is an $n \times n$ symmetric matrix, I is the $n \times n$ identity matrix. The value of Σ can be obtained from the previous section. More specifically, $\text{Var}_k[\check{D}_{k+1}]$ is the (1,1) entry of the matrix $\text{Cov}[\check{y}_{k+1}, \check{y}_{k+1}]$ in equation (A-17), $\text{Cov}[\check{D}_{k+1}, y_{k+1}]$ is given by the first column of the matrix $\text{Cov}[y_{k+1}, \check{y}_{k+1}]$ in equation (A-17), and $\text{Var}_k[y_{k+1}]$ is given in equation (A-16).

Define α , $\bar{A}$, B , and C as

$$\begin{aligned}
\bar{A} = & \psi_0 + \psi_W e^{rh} (W_{n,k} - hc_{n,k} - S_{n,k} P_k) + \psi_W S_{n,k} \left(E_{kh}^n [\check{D}_{k+1}] + \frac{E_{kh}^n [D_{k+1}]}{r + \alpha_D} \right. \\
& \left. + \frac{C_G \sigma_G \Omega^{1/2} (\tau_H^{1/2} + (N-1) \tau_L^{1/2})}{N(r + \alpha_D)(r + \alpha_G)} (E_{kh}^n [H_{n,k+1}^c] + (N-1) E_{kh}^n [H_{-n,k+1}^c]) \right) \\
& + \frac{1}{2} \psi_{nn} (E_{kh}^n [H_{n,k+1}^c])^2 + \frac{1}{2} \psi_{xx} (E_{kh}^n [H_{-n,k+1}^c])^2 + \psi_{nx} H_{n,k+1}^c H_{-n,k+1}^c, \\
\alpha = & -1, \quad B = \psi_W \psi_B S_{n,k} + \varphi_B, \quad C = \begin{bmatrix} 0_{2 \times 2} & 0_{2 \times 2} \\ 0_{2 \times 2} & c_{2 \times 2} \end{bmatrix}, \quad \text{where} \\
c_{2 \times 2} = & \begin{bmatrix} \psi_{nn} & \psi_{nx} \\ \psi_{nx} & \psi_{xx} \end{bmatrix}, \quad \psi_B = \psi_{B1} + \frac{C_G \sigma_G \Omega^{1/2} (\tau_H^{1/2} + (N-1) \tau_L^{1/2})}{N(r + \alpha_D)(r + \alpha_G)} \psi_{B2}, \\
\psi_{B1} = & \left[\frac{1}{r + \alpha_D}, 1, 0, 0 \right]', \quad \psi_{B2} = [0, 0, 1, N-1]', \\
\varphi_B = & \left[0, 0, \psi_{nn} E_{kh}^n [H_{n,k+1}^c] + \psi_{nx} E_{kh}^n [H_{-n,k+1}^c], \psi_{xx} E_{kh}^n [H_{-n,k+1}^c] + \psi_{nx} E_{kh}^n [H_{n,k+1}^c] \right]'.
\end{aligned} \tag{A-24}$$

Taking the first-order condition with respect to $c_{n,k}$ in the HJB equation (A-23) yields

$$c_{n,k}^* = -\frac{1 - e^{-rh}}{Arh} \left((r - \rho)h + \ln \left(\frac{\psi_W (1 - e^{-rh})}{Arh} E_{kh}^n V_{k+1} \right) \right). \tag{A-25}$$

Substituting (A-25) into the HJB equation (A-23) yields

$$V_k = e^{-\rho h} \left(1 - \frac{\psi_W (e^{rh} - 1)}{Ar} \right) E_{kh}^n [V_{k+1}]. \tag{A-26}$$

Taking the first-order condition with respect to $S_{n,k}$ yields

$$S_{n,k}^* = \frac{E_{kh}^n [P_{k+1} + \check{D}_{k+1}] - e^{rh} P_k + \varphi_B' \Sigma (I - C \Sigma)^{-1} \psi_B}{-\psi_W \psi_B' \Sigma (I - C \Sigma)^{-1} \psi_B}. \tag{A-27}$$

It can be shown that optimal quantity $S_{n,k}^*$ is linear in the state variables $H_{n,k}^c$ and $H_{-n,k}^c$. Define constants c_{d1} , c_{d2} , c_{n1} , c_{x1} , c_{n2} , and c_{x2} by

$$\begin{aligned}
c_{d1} &:= \frac{\sigma_G \Omega^{1/2} (e^{-\alpha_G h} - e^{-\alpha_D h})}{\alpha_D - \alpha_G}, \quad c_{d2} := \frac{\sigma_G \Omega^{1/2} ((r + \alpha_G) e^{-(r + \alpha_D)h} - (r + \alpha_D) e^{-(r + \alpha_G)h} + \alpha_D - \alpha_G)}{(\alpha_D - \alpha_G)(r + \alpha_D)(r + \alpha_G)}, \\
c_{n1} &:= \frac{e^{-(\alpha_G + \tau)h} (e^{\tau h} (\tau_H^{1/2} (\tau_0 + \tau_H) + (N-1) \tau_H \tau_L^{1/2}) + (N-1) \tau_L^{1/2} (\tau_0 + \tau_H^{1/2} \tau_L^{1/2} + (N-1) \tau_L))}{(\tau_H^{1/2} + (N-1) \tau_L^{1/2}) \tau},
\end{aligned}$$

$$c_{n2} := \frac{e^{-(\alpha_G + \tau)h} (e^{\tau h} - 1) \tau_H^{1/2} (\tau_0 + \tau_H^{1/2} \tau_L^{1/2} + (N-1) \tau_L)}{(\tau_H^{1/2} + (N-1) \tau_L^{1/2}) \tau}, \quad c_{x1} := \frac{e^{-(\alpha_G + \tau)h} (e^{\tau h} - 1) (N-1) \tau_L^{1/2} (\tau_0 + (N-1) \tau_H^{1/2} \tau_L^{1/2} + \tau_H)}{(\tau_H^{1/2} + (N-1) \tau_L^{1/2}) \tau},$$

$$c_{x2} := \frac{e^{-(\alpha_G + \tau)h} (e^{\tau h} (N-1) \tau_L^{1/2} (\tau_0 + \tau_H^{1/2} \tau_L^{1/2} + (N-1) \tau_L) + \tau_H^{1/2} (\tau_0 + \tau_H) + (N-1) \tau_H \tau_L^{1/2})}{(\tau_H^{1/2} + (N-1) \tau_L^{1/2}) \tau}.$$

Then φ_B , defined in equation (A-24), can be written as $\varphi_B = \varphi_{B1} H_{n,k}^c + \varphi_{B2} H_{-n,k}^c$, where $\varphi_{B1} := [0, 0, \psi_{nn} c_{n1} + \psi_{nx} c_{n2}, \psi_{xx} c_{n2} + \psi_{nx} c_{n1}]'$, and $\varphi_{B2} := [0, 0, \psi_{nn} c_{x1} + \psi_{nx} c_{x2}, \psi_{xx} c_{x2} + \psi_{nx} c_{x1}]'$. The market clearing condition $\sum_{n=1}^N S_{n,k}^* = 0$ and equation (A-27) imply

$$C_G = \frac{e^{-\alpha_G h} - e^{rh} - \frac{(r + \alpha_D)(r + \alpha_G)}{\sigma_G \Omega^{1/2} (\tau_H^{1/2} + (N-1) \tau_L^{1/2})} (\varphi'_{B1} + \varphi'_{B2}) \Sigma(I - C\Sigma)^{-1} \psi_{B1}}{\frac{N \tau_0 + (\tau_H^{1/2} + (N-1) \tau_L^{1/2})^2}{N\tau} e^{-\alpha_G h} - e^{rh} + \frac{(N-1) (\tau_H^{1/2} - \tau_L^{1/2})^2}{N\tau} e^{-(\alpha_G + \tau)h} + \frac{1}{N} (\varphi'_{B1} + \varphi'_{B2}) \Sigma(I - C\Sigma)^{-1} \psi_{B2}}. \quad (A-28)$$

Then, from equations (A-27) and (A-28), the optimal inventory for trader n is given by

$$S_{n,k}^* = C_L (H_{n,k}^c - H_{-n,k}^c), \quad (A-29)$$

where the constant C_L is defined as

$$C_L = \frac{1}{r A N \tau (r + \alpha_D) (r + \alpha_G) \psi'_B \Sigma(I - C\Sigma)^{-1} \psi_B} \left(N \tau \left(\sigma_G \Omega^{1/2} \tau_H^{1/2} (e^{rh} - e^{-\alpha_G h}) + (r + \alpha_D) (r + \alpha_G) \varphi'_{B1} \Sigma(I - C\Sigma)^{-1} \psi_{B1} \right) \right. \\ \left. + C_G \sigma_G \Omega^{1/2} \left(\tau \left(\tau_H^{1/2} + (N-1) \tau_L^{1/2} \right) \varphi'_{B1} \Sigma(I - C\Sigma)^{-1} \psi_{B2} + e^{-\alpha_G h} \tau_H^{1/2} \left(N \tau_0 + \left(\tau_H^{1/2} + (N-1) \tau_L^{1/2} \right)^2 \right) \right. \right. \\ \left. \left. - e^{rh} \left(\tau_H^{1/2} + (N-1) \tau_L^{1/2} \right) \tau - e^{-(\alpha_G + \tau)h} (N-1) \left(\tau_H^{1/2} - \tau_L^{1/2} \right) \left(\tau_0 + \tau_H^{1/2} \tau_L^{1/2} + (N-1) \tau_L \right) \right) \right). \quad (A-30)$$

Equations (A-26) and (A-22) imply

$$\ln(-E_{kh}^n V_{k+1}) = \psi_0 + \psi_W W_{n,k} + \frac{1}{2} \psi_{nn} (H_{n,k}^c)^2 + \frac{1}{2} \psi_{xx} (H_{-n,k}^c)^2 + \psi_{nx} H_{n,k}^c H_{-n,k}^c + \rho h - \ln \left(1 - \frac{\psi_W (e^{rh} - 1)}{Ar} \right). \quad (A-31)$$

Substituting (29), (A-16), (A-18), value function (A-22), (A-25), (A-29), and (A-31) into the HJB equation (A-23) and setting the constant term and the coefficients of $W_{n,k}$, $(H_{n,k}^c)^2$, $(H_{-n,k}^c)^2$, and $H_{n,k}^c H_{-n,k}^c$ to be zero, we obtain five equations, from which we can find five unknown parameters ψ_0 , ψ_W , ψ_{nn} , ψ_{nx} , and ψ_{xx} . By setting the constant term and coefficient of $W_{n,k}$ to be zero, we obtain

$$\psi_W = -r A, \quad \psi_0 = \frac{(r - \rho)h - (e^{rh} - 1) \ln \frac{1 - e^{-rh}}{h} - \ln \sqrt{|I - C\Sigma|}}{e^{rh} - 1}. \quad (A-32)$$

By setting the coefficients of $(H_{n,k}^c)^2$, $(H_{-n,k}^c)^2$ and $H_{n,k}^c H_{-n,k}^c$ to be zero, we obtain three polynomial equations in the three unknowns ψ_{nn} , ψ_{xx} , and ψ_{nx} . These three equations in three unknowns can be written as the following system of equations:

$$\begin{aligned} 0 = & -\frac{1}{2} e^{rh} \psi_{nn} + r AC_L C_G \left(e^{rh} - c_{n1} - (N-1)c_{n2} \right) \frac{\sigma_G \Omega^{1/2} (\tau_H^{1/2} + (N-1)\tau_L^{1/2})}{N(r+\alpha_D)(r+\alpha_G)} \\ & - r AC_L \left(e^{rh} c_{d2} + \frac{c_{d1}}{r+\alpha_D} \right) \tau_H^{1/2} + \frac{1}{2} \psi_{nn} c_{n1}^2 + \frac{1}{2} \psi_{xx} c_{n2}^2 + \psi_{nx} c_{n1} c_{n2} \\ & + \frac{1}{2} r^2 A^2 C_L^2 \psi'_B \Sigma (I - C\Sigma)^{-1} \psi_B - r AC_L \psi'_B \Sigma (I - C\Sigma)^{-1} \varphi_{B1} + \frac{1}{2} \varphi'_{B1} \Sigma (I - C\Sigma)^{-1} \varphi_{B1}, \end{aligned} \quad (A-33)$$

$$\begin{aligned} 0 = & -\frac{1}{2} e^{rh} \psi_{xx} + r AC_L C_G \left(-e^{rh} (N-1) + c_{x1} + (N-1)c_{x2} \right) \frac{\sigma_G \Omega^{1/2} (\tau_H^{1/2} + (N-1)\tau_L^{1/2})}{N(r+\alpha_D)(r+\alpha_G)} \\ & + r AC_L \left(e^{rh} c_{d2} + \frac{c_{d1}}{r+\alpha_D} \right) (N-1) \tau_L^{1/2} + \frac{1}{2} \psi_{xx} c_{x2}^2 + \psi_{nx} c_{x1} c_{x2} \\ & + \frac{1}{2} r^2 A^2 C_L^2 \psi'_B \Sigma (I - C\Sigma)^{-1} \psi_B + r AC_L \psi'_B \Sigma (I - C\Sigma)^{-1} \varphi_{B2} + \frac{1}{2} \varphi'_{B2} \Sigma (I - C\Sigma)^{-1} \varphi_{B2}, \\ 0 = & e^{rh} \psi_{nx} + r AC_L C_G \left(e^{rh} (N-2) - c_{x1} - (N-1)c_{x2} + c_{n1} + (N-1)c_{n2} \right) \frac{\sigma_G \Omega^{1/2} (\tau_H^{1/2} + (N-1)\tau_L^{1/2})}{N(r+\alpha_D)(r+\alpha_G)} \\ & - r AC_L \left(e^{rh} c_{d2} + \frac{c_{d1}}{r+\alpha_D} \right) \left((N-1) \tau_L^{1/2} - \tau_H^{1/2} \right) + \psi_{nn} c_{n1} c_{x1} + \psi_{xx} c_{n2} c_{x2} + \psi_{nx} (c_{n1} c_{x2} + c_{x1} c_{n2}) \\ & - r^2 A^2 C_L^2 \psi'_B \Sigma (I - C\Sigma)^{-1} \psi_B - r AC_L \psi'_B \Sigma (I - C\Sigma)^{-1} (\varphi_{B2} - \varphi_{B1}) + \varphi'_{B1} \Sigma (I - C\Sigma)^{-1} \varphi_{B2}. \end{aligned} \quad (A-34)$$

To summarize, optimal consumption is defined in (A-25), the optimal strategy is defined in (A-29), and the endogenous coefficient C_L is defined in (A-30). The equilibrium price is defined in (A-18), and the endogenous coefficient C_G is defined in (A-28). The quadratic value function is defined by five parameters. Parameters ψ_W and ψ_0 are presented in (A-32). Parameters ψ_{nn} , ψ_{nx} , ψ_{xx} are solved numerically from the system of the three equations (A-33)–(A-34). This concludes the proof of Theorem 4.

A.7.3 The Case with Myopic Conditional Mean-Variance Optimizers

Trader n 's myopic conditional mean-variance optimal holding at time kh is given in equation (39). Similar to the proof of the general dynamic model case (equation (A-27)), it can be shown that optimal quantity $\check{S}_{n,k}$ is linear in the state variables $H_{n,k}^c$ and $H_{-n,k}^c$. The market clearing condition $\sum_{n=1}^N \check{S}_{n,k} = 0$ and equation (39) imply

$$\check{C}_G = \frac{e^{-\alpha_G h} - e^{rh}}{\frac{N\tau_0 + \left(\tau_H^{1/2} + (N-1)\tau_L^{1/2}\right)^2}{N\tau} e^{-\alpha_G h} - e^{rh} + \frac{(N-1)\left(\tau_H^{1/2} - \tau_L^{1/2}\right)^2}{N\tau} e^{-(\alpha_G + \tau)h}},$$

which can be simplified as equation (41). This completes the proof of Theorem 3.

A.8 Proof of Proposition 4

The endogenous parameter $\check{C}_G$ is defined in equation (41). From equation (41), we can see that a common prior ($\tau_H = \tau_L$) implies $\check{C}_G = 1$. Relative overconfidence ($\tau_H > \tau_L$) implies $0 < \check{C}_G < 1$. Fixing the total precision τ , then $\check{C}_G$ decreases in the disagreement τ_H/τ_L . Using Taylor expansion, it can be shown that $\frac{1-e^{-\tau h}}{e^{(r+\alpha_G)h}-1}$ decreases in trading interval h . It follows that $\check{C}_G$ increases in the trading interval h .

A.9 Proof of Proposition 5

Assuming $\tau_H = \tau_L$, then clearly $\psi_{nn} = \psi_{nx} = \psi_{xx} = 0$ solves the three equations (A-33)–(A-34). We also get $C_G = 1$ and $C_L = 0$ from equations (A-28) and (A-30). There is no trading. If $\tau_H = \tau_L$, then $C_J = 1$ in equation (48); if $\tau_H > \tau_L$, then Jensen's inequality implies $0 < C_J < 1$.

In the case with relative overconfidence of $\tau_H > \tau_L$, information cannot have negative value in the value function (A-22) since traders can always ignore it. Therefore, the 2×2 matrix

$$\begin{bmatrix} \psi_{nn} & \psi_{nx} \\ \psi_{nx} & \psi_{xx} \end{bmatrix}$$

must be negative semi-definite. This implies $\psi_{nn} \leq 0$, $\psi_{xx} \leq 0$, and $\psi_{nx}^2 \leq \psi_{nn}\psi_{xx}$. It follows that $\psi_{nn} + \psi_{xx} + 2\psi_{nx} \leq 0$. In the continuous-time model ($h \rightarrow 0$), we can show

$$C_G = \frac{N(r+\alpha_G) \left(\sigma_G \Omega^{1/2} + \frac{\sigma_D \hat{a} (\psi_{nn} + \psi_{xx} + 2\psi_{nx})}{\tau_H^{1/2} + (N-1)\tau_L^{1/2}} \right)}{\sigma_G \Omega^{1/2} \left(N(r+\alpha_G) + (N-1) \left(\tau_H^{1/2} - \tau_L^{1/2} \right)^2 - (1+N\hat{a}^2) (\psi_{nn} + \psi_{xx} + 2\psi_{nx}) \right)}.$$

Then it can be shown that $\lim_{h \rightarrow 0} C_G \leq \lim_{h \rightarrow 0} \check{C}_G < 1$. This concludes the proof.

A.10 Proof of Proposition 6

Let a vector $[\psi_{nn}^*, \psi_{nx}^*, \psi_{xx}^*]$ be a solution to the system (A-33)–(A-34) for exogenous parameters $A, \sigma_D, \sigma_G, r, \alpha_G, \alpha_D, \tau_0, \tau_L$, and τ_H . If risk aversion is rescaled by factor F from A to A/F and other exogenous parameters are kept unchanged, then it is straightforward to show that the vector $[\psi_{nn}^*, \psi_{nx}^*, \psi_{xx}^*]$ is still the solution to the system (A-33)–(A-34). From equations (A-28) and (A-30), it then follows that C_L changes to $C_L F$, but C_G remains the same.

A.11 Proof of Theorem 5

Since traders use models with correct structure but with possibly incorrect parameters, we first calculate the empirically correct estimate of the growth rate and then derive the holding-period excess return.

A.11.1 Inference under Correct Beliefs

The growth rate defined in equation (22) needs to be rewritten using the true parameters,

$$dG^*(t) := -\hat{\alpha}_G G^*(t) dt + \hat{\sigma}_G dB_G(t).$$

Similarly, we rewrite the process for signals $dI_n(t)$ and $dI_0(t)$ defined in equations (24) and (25) using the true parameters,

$$\begin{aligned} dI_n(t) &:= \hat{\tau}_I^{1/2} \frac{G^*(t)}{\hat{\sigma}_G \hat{\Omega}^{1/2}} dt + d\hat{B}_n(t), \quad n = 1, \dots, N, \quad dI_0(t) := \hat{\tau}_0^{1/2} \frac{G^*(t)}{\hat{\sigma}_G \hat{\Omega}^{1/2}} dt + dB_0(t), \\ \text{with } \hat{\tau}_0 &:= \frac{\hat{\Omega} \hat{\sigma}_G^2}{\sigma_D^2} \quad \text{and} \quad d\hat{B}_n(t) = dB_n(t) + \left(\frac{\tau_n^{1/2}}{\sigma_G \Omega^{1/2}} - \frac{\hat{\tau}_n^{1/2}}{\hat{\sigma}_G \hat{\Omega}^{1/2}} \right) G^*(t) dt, \end{aligned} \quad (\text{A-35})$$

and $dB_G(t)$, $dB_0(t)$, $d\hat{B}_1(t)$, $\dots$, $d\hat{B}_N(t)$ are independent Brownian motions.

The sufficient statistic under the correct parameters is defined as

$$\hat{H}_n(t) := \int_{u=-\infty}^t e^{-(\hat{\alpha}_G + \hat{\tau})(t-u)} dI_n(u), \quad n = 0, 1, \dots, N.$$

The sufficient statistic $\hat{H}_n(t)$ and $H_n(t)$ relate to each other as follows:

$$\hat{H}_n(t) = H_n(t) + (\alpha_G + \tau - \hat{\alpha}_G - \hat{\tau}) \int_{u=-\infty}^t e^{-(\hat{\alpha}_G + \hat{\tau})(t-u)} H_n(u) du. \quad (\text{A-36})$$

If traders use the empirically correct parameters ($\alpha_G = \hat{\alpha}_G$), and empirically correct total precisions of signals ($\tau = \hat{\tau}$), then we obtain $\hat{H}_n(t) = H_n(t)$. If $\alpha_G + \tau \neq \hat{\alpha}_G + \hat{\tau}$, then $\hat{H}_n(t) \neq H_n(t)$. Define the aggregate sufficient statistic as the linear combination of $\hat{H}_n(t)$, $n = 0, 1, \dots, N$,

$$\hat{H}(t) := \hat{\tau}_0^{1/2} \hat{H}_0(t) + \sum_{n=1}^N \hat{\tau}_I^{1/2} \hat{H}_n(t).$$

We then redo the calculation of Lemma 1 in Section A.6 to derive the Stratonovich–Kalman–Bucy filtering formulas using the empirically correct parameters. We obtain the empirically

correct estimate of the growth rate at $t = kh$: $\hat{G}_k := \hat{E}_{kh} [G^*(kh)] = \sigma_G \hat{\Omega}^{1/2} \hat{H}_k$, and the steady-state error variance

$$\hat{\Omega} := \text{Var} \left[\frac{G^*(t) - \hat{G}(t)}{\hat{\sigma}_G} \right] = \frac{1}{2\hat{\alpha}_G + \hat{\tau}}.$$

A.11.2 The Equilibrium Return Dynamics

Plugging equation (38) into the continuous version of equation (46) yields price $P(t)$ at time t :

$$P(t) = \frac{D(t)}{r + \alpha_D} + C_G \frac{\sigma_G \Omega^{1/2}}{(r + \alpha_D)(r + \alpha_G)} H(t). \quad (\text{A-37})$$

The price depends on dividend $D(t)$ and traders' sufficient statistics $H(t)$ about growth rate; it does not depend on empirically correct sufficient statistics $\hat{H}(t)$.

The dynamics of state variables is determined by empirically correct parameters. The return dynamics thus also depend on both traders' sufficient statistics $H(t)$ and empirically correct sufficient statistics $\hat{H}(t)$. Specially, when we calculate $dP(t)$ using equation (A-37), we plug in $dH_n(t)$ using equation (33), and plug in the correct empirical specification of the dynamics of $dI_n(t)$ from equation (A-35) and the correct estimate $\hat{G}(t)$ from equation (51).

We can show that the equilibrium instantaneous return process is a linear combination of the two statistics $H(t)$ and $\hat{H}(t)$,

$$dP(t) + D(t) dt - rP(t) dt = (b\hat{H}(t) - aH(t)) dt + d\hat{B}_r(t), \quad (\text{A-38})$$

where constants a , b , and $d\hat{B}_r(t)$ are defined as follows:

$$a := \frac{\sigma_G C_G \Omega^{1/2}}{(r + \alpha_D)(r + \alpha_G)} (\alpha_G + r + \tau), \quad b := \frac{\sigma_G \hat{\Omega}^{1/2}}{r + \alpha_D} + \frac{\sigma_G C_G \Omega^{1/2}}{(r + \alpha_D)(r + \alpha_G)} \left(\tau_0^{1/2} \hat{\tau}_0^{1/2} + N \tau_I^{1/2} \hat{\tau}_I^{1/2} \right),$$

$$d\hat{B}_r(t) := \frac{\sigma_D}{r + \alpha_D} dB_0^*(t) + \frac{\sigma_G C_G \Omega^{1/2}}{(r + \alpha_D)(r + \alpha_G)} \left(\tau_0^{1/2} dB_0^*(t) + \tau_I^{1/2} N d\bar{B}^*(t) \right).$$

The diffusion term $d\hat{B}_r(t)$ depends on the processes $d\bar{B}^*(t)$ and $dB_0^*(t)$:

$$\begin{aligned} d\bar{B}^*(t) &:= \hat{\tau}_I^{1/2} (\sigma_G \hat{\Omega}^{1/2})^{-1} (G^*(t) - \hat{G}(t)) dt + \frac{1}{N} \sum_{n=1}^N d\hat{B}_n(t), \\ dB_0^*(t) &:= \hat{\tau}_0^{1/2} (\sigma_G \hat{\Omega}^{1/2})^{-1} (G^*(t) - \hat{G}(t)) dt + dB_0(t). \end{aligned}$$

Since $\hat{G}(t)$ is the best estimate of $G^*(t)$, both $d\bar{B}^*(t)$ and dB_0 are Brownian motions under

empirically correct beliefs, with variances $\text{Var}[dB_0^*(t)] = dt$ and $\text{Var}[d\bar{B}^*(t)] = 1/N$. So the instantaneous variance of the excess return is given by

$$\hat{\text{Var}}\left[\frac{d\hat{B}_r(t)}{dt^{1/2}}\right] = \left(\frac{\sigma_D}{r+\alpha_D} + \frac{\sigma_G \Omega^{1/2} C_G \tau_0^{1/2}}{(r+\alpha_D)(r+\alpha_G)}\right)^2 + \frac{(\sigma_G \Omega^{1/2} C_G)^2 N \tau_I}{(r+\alpha_D)^2 (r+\alpha_G)^2}. \quad (\text{A-39})$$

Define a continuous 2-vector stochastic process $y_H(t) = [H(t), \hat{H}(t)]'$. Using the definitions of $H(t)$ and $\hat{H}(t)$ in equations (37) and (50), $y_H(t)$ can be shown to satisfy the linear stochastic differential equation

$$dy_H(t) = K_H y_H(t) dt + C_H dZ_H(t), \quad (\text{A-40})$$

where K_H is a 2×2 matrix and C_H is a 2×2 matrix given by

$$K_H = \begin{bmatrix} -\alpha_G - \tau & \hat{\tau}_0^{1/2} \tau_0^{1/2} + N \hat{\tau}_I^{1/2} \tau_I^{1/2} \\ 0 & -\hat{\alpha}_G \end{bmatrix}, \quad C_H = \begin{bmatrix} \tau_0^{1/2} & \sqrt{N} \tau_I^{1/2} \\ \hat{\tau}_0^{1/2} & \sqrt{N} \hat{\tau}_I^{1/2} \end{bmatrix}.$$

Under empirically correct beliefs, the vector $dZ_H(t) = [dB_0^*(t), \sqrt{N} d\bar{B}^*(t)]'$ is a 2×1 -dimensional Brownian motion, with $\text{Var}[dB_0^*(t)] = dt$ and $\text{Var}[d\bar{B}^*(t)] = dt/N$. Integrating equation (A-38) over time yields the holding period return over f periods from $t = kh$ to $t + T = (k + f)h$:

$$R_{k,k+f} = \int_{u=kh}^{kh+T} e^{r(kh+T-u)} (dP(u) + D(u) du - rP(u) du) = P_{k+f} - e^{rT} P_k + e^{rT} \int_{u=kh}^{kh+T} e^{-r(u-kh)} D(u) du. \quad (\text{A-41})$$

Using equation (A-40), we can obtain recursive formulas for the stochastic vector $y_H(u) = [H(u), \hat{H}(u)]'$ for time $u \geq kh$ as a function of $y_H(kh) = [H_k, \hat{H}_k]'$. Plugging these recursive formulas into equation (A-41), we obtain the cumulative holding period return $R_{k,k+f}$ over period $T = fh$ between time kh and $(k + f)h$ as a linear function of H_k and $\hat{H}_k$:

$$R_{k,k+f} = \zeta_2(T) \hat{H}_k - \zeta_1(T) H_k + \bar{B}_{k,k+f}. \quad (\text{A-42})$$

The coefficients $\zeta_1(T) > 0$ and $\zeta_2(T) > 0$ in the equation above are defined as

$$\begin{aligned} \zeta_1(T) &:= \frac{\sigma_G C_G \Omega^{1/2}}{(r + \alpha_D)(r + \alpha_G)} e^{rT} (1 - e^{-(r + \alpha_G + \tau)T}) > 0, \\ \zeta_2(T) &:= \frac{e^{rT} - e^{-\hat{\alpha}_G T}}{r + \hat{\alpha}_G} \frac{\hat{\sigma}_G \hat{\Omega}^{1/2}}{r + \alpha_D} + \frac{\sigma_G C_G \Omega^{1/2} (\tau_0^{1/2} \hat{\tau}_0^{1/2} + N \hat{\tau}_I^{1/2} \tau_I^{1/2})}{(\tau + \alpha_G - \hat{\alpha}_G)(r + \alpha_G)(r + \alpha_D)} (e^{-\hat{\alpha}_G T} - e^{-(\alpha_G + \tau)T}) > 0. \end{aligned} \quad (\text{A-43})$$

and the Brownian motion term $\bar{B}_{k,k+f}$ is defined as

$$\bar{B}_{k,k+f} := \int_{s=kh}^{kh+T} \int_{u=s}^{kh+T} [-a, b] e^{r(kh+T-u)+K_H(u-s)} C_H du dZ_H(s) + \int_{s=kh}^{kh+T} e^{r(kh+T-s)} d\hat{B}_r(s).$$

A.11.3 Proof of Theorem 5

If traders have correct model parameters $\hat{\alpha}_G = \alpha_G$, $\hat{\sigma}_G = \sigma_G$, and $\hat{\tau} = \tau$, then $\zeta_1(T)$ and $\zeta_2(T)$ defined in equation (A-43) can be written as

$$\begin{aligned} \zeta_1(T) &= \frac{C_G \sigma_G \Omega^{1/2}}{(r + \alpha_G)(r + \alpha_D)} (e^{rT} - e^{-(\alpha_G + \tau)T}), \\ \zeta_2(T) &= \frac{\sigma_G \Omega^{1/2} e^{-(\alpha_G + \tau)T}}{(r + \alpha_G)(r + \alpha_D) \tau} \left((e^{(r + \alpha_G + \tau)T} - e^{\tau T}) \tau + C_G (\tau_0 + N \hat{\tau}_I^{1/2} \tau_I^{1/2}) (e^{\tau T} - 1) \right). \end{aligned} \quad (\text{A-44})$$

Since both $H(t)$ and $\hat{H}(t)$ can be recovered from the history of prices $P(t)$ and dividends $D(t)$, the expected holding-period return $\hat{E}_{kh}[R_{k,k+f}]$ can be expressed as a linear combination of P_k , D_k , and $H_{0,k}$, as in equation (52), with corresponding coefficients $\beta_0(T)$ and $\beta_1(T)$:

$$\beta_1(T) := \left(\zeta_2(T) \frac{\hat{\tau}_I^{1/2}}{\tau_I^{1/2}} - \zeta_1(T) \right) \frac{(r + \alpha_D)(r + \alpha_G)}{C_G \sigma_G \Omega^{1/2}}, \quad \beta_0(T) := \zeta_2(T) \left(\frac{\hat{\tau}_I^{1/2}}{\tau_I^{1/2}} - 1 \right) \tau_0^{1/2}. \quad (\text{A-45})$$

A common prior ($\tau_H = \tau_L$) implies $C_G = C_J = 1$ and $\hat{\tau}_I^{1/2} = \tau_I^{1/2} = 0$ and, thus, $\beta_0(T) = 0$ and $\beta_1(T) = 0$. Direct computations show that relative overconfidence ($\tau_H > \tau_L$) implies $\beta_1(T) > 0$, $d\beta_1(T)/dT > 0$, and $\beta_0(T) > 0$. Also, it can be shown that the term $\zeta_2(T) \hat{\tau}_I^{1/2} - \zeta_1(T) \tau_I^{1/2}$ in the definition of $\beta_1(T)$ in equation (A-45) can be written as

$$\begin{aligned} \zeta_2(T) \hat{\tau}_I^{1/2} - \zeta_1(T) \tau_I^{1/2} &= \frac{\sigma_G \Omega^{1/2} e^{rT}}{(r + \alpha_D)(r + \alpha_G)} \left(\hat{\tau}_I^{1/2} (1 - C_G) (1 - e^{-(r + \alpha_G)T}) \right. \\ &\quad \left. + C_G \frac{\hat{\tau}_I^{1/2} - \tau_I^{1/2}}{\tau} (N \hat{\tau}_I (1 - e^{-(r + \alpha_G)T}) + \tau_0 (1 - e^{-(r + \alpha_G + \tau)T})) \right). \end{aligned} \quad (\text{A-46})$$

Thus, the coefficient $\beta_1(T) > 0$ can be decomposed into two terms. The first term in large parenthesis with $1 - C_G > 0$ results from the price dampening effect of the Keynesian beauty contest, and the second term with $\hat{\tau}_I^{1/2} - \tau_I^{1/2} > 0$ results from the price dampening effect of $C_J < 1$. This

concludes the proof of Theorem 5.

A.11.4 The Expected Cumulative Holding-Period Return with $\alpha_G \neq \hat{\alpha}_G$ and $\hat{\tau} \neq \tau$

Both $H(t)$ and $\hat{H}(t)$ can be recovered from the history of prices $P(t)$ and dividends $D(t)$. If traders have incorrect model parameters and total precisions of signals ($\alpha_G \neq \hat{\alpha}_G$ and $\hat{\tau} \neq \tau$), then the expected holding-period excess return, $\hat{E}_t[R(t, t+T)]$ can be expressed as in equation (53). The coefficients $\zeta_1(T) > 0$ and $\zeta_2(T) > 0$ are defined in equation (A-43). We study the return autocorrelations for equal leads and lags, $T = T_l = T_f$.

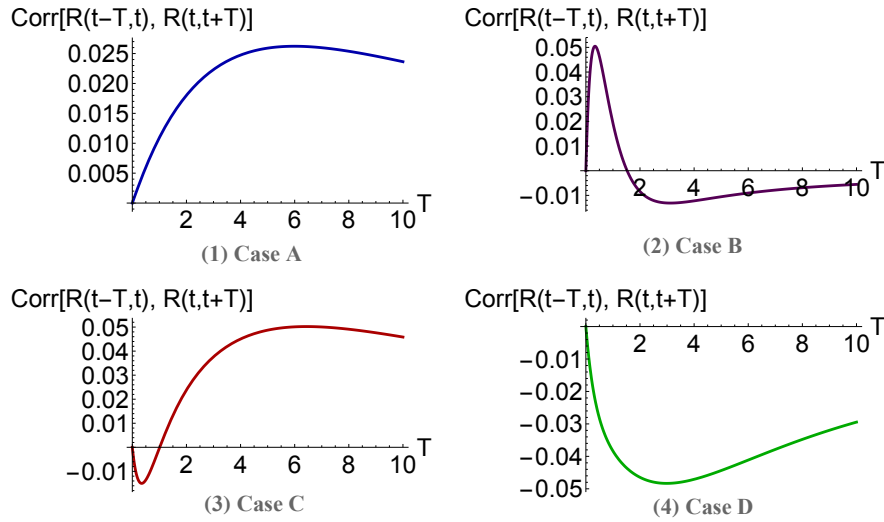


Figure A-1: The four panels illustrate different patterns of positive and negative correlation, $\text{Corr}[R(t-T, t), R(t, t+T)]$, obtained over different holding periods T .

Figure A-1 plots the autocorrelations $\text{Corr}[R(t-T, t), R(t, t+T)]$ for different horizons T when traders have incorrect beliefs about the total precision of signals ($\hat{\tau} \neq \tau$) and model parameters ($\hat{\alpha}_G \neq \alpha_G$).^{A.1} There are four possible patterns: (1) only momentum, (2) first momentum and then mean-reversion, (3) first mean-reversion and then momentum, and (4) only mean-reversion for four specific combinations of different parameter values.

^{A.1}The default parameter values of $r, A, N, \alpha_D, \sigma_D, \alpha_G, \sigma_G, \tau_H$, and τ_L are given in Table 2 in Section 4. In Panel (1), $\hat{\tau} = \tau = 2$. In Panel (2), $\hat{\tau} = 1.6 < \tau$, $\hat{\alpha}_G = 1 > \alpha_G$, and $\hat{\sigma}_G = 0.5 > \sigma_G$. In Panel (3), $\hat{\tau} = 1.6 < \tau$. In Panel (4), $\hat{\alpha}_G = 0.4 > \alpha_G$ and $\hat{\tau} = 1.6 < \tau$.

B Online Appendix

Section B.1 shows that the static dampening, related to $C_J < 1$, shows up in a one-period model. Section B.2 calculates each trader's expectations about how his own valuation, the average valuation of other traders, and the market price evolve over time. Section B.3 discusses how our calibration results are affected by parameter values. Section B.4 presents empirical results that time-series momentum tends to be more pronounced in securities with higher trading volume. Section B.5 presents detailed formulas for the autocorrelation of the holding-period return. Section B.6 presents a two-period model with more general signal structure where traders might potentially disagree about both the coefficient on the fundamental value (drift term) and the coefficient on the noise (diffusion term) in signals. We show that return momentum arises when disagreement on the drift term dominates disagreement on the diffusion term. Section B.7 considers a special case where static dampening does not occur and only dynamic dampening arises; we show that returns still exhibit momentum due to dynamic price dampening. Section B.8 shows, using traders' beliefs, that dynamic price dampening leads to time-series return momentum, and momentum is greater when traders can trade more frequently.

B.1 A One-Period Model

A risky asset with random liquidation value $v \sim \mathcal{N}(0, 1/\tau_v)$ is traded for a safe numeraire asset. The total supply of the risky asset is zero. Traders observe signals about the normalized liquidation value $\tau_v^{1/2} v$. Each trader n observes a private signal $i_n := \tau_n^{1/2} (\tau_v^{1/2} v) + e_n$ with $e_n \sim \mathcal{N}(0, 1)$. The asset payoff v and N private signal errors $e_1, \dots, e_N$ are independently distributed.

Traders agree to disagree about the precisions of private signals τ_n . Each trader is “relatively overconfident,” believing his own signal has a high precision $\tau_n = \tau_H$ and other traders' signals have low precision $\tau_m = \tau_L$ for $m \neq n$, with $\tau_H > \tau_L \geq 0$.

Let $E^n[\dots]$ and $\text{Var}^n[\dots]$ denote trader n 's expectation and variance operators conditional on all signals $i_1, \dots, i_N$. Define “total precision” τ by

$$\tau := (\text{Var}^n[v])^{-1} = \tau_v (1 + \tau_H + (N-1)\tau_L).$$

The projection theorem for jointly normally distributed random variables implies

$$E^n[v] = \frac{\tau_v^{1/2}}{\tau} \left(\tau_H^{1/2} i_n + (N-1)\tau_L^{1/2} i_{-n} \right), \quad \text{where} \quad i_{-n} := \frac{1}{N-1} \sum_{m=1, m \neq n}^N i_m.$$

Trader n chooses the optimal quantity S_n to hold by maximizing his expected exponential utility of terminal wealth:

$$\mathbb{E}^n \left[-e^{-A W_n} \right], \quad \text{where} \quad W_n := (\nu - P) S_n.$$

The first-order condition with respect to S_n yields

$$S_n = \frac{\mathbb{E}^n[\nu] - P}{A \text{Var}^n[\nu]} = \frac{1}{A} \left(\tau_v^{1/2} \left(\tau_H^{1/2} i_n + (N-1) \tau_L^{1/2} i_{-n} \right) - P \tau \right). \quad (\text{B-1})$$

The market-clearing condition $\sum_{n=1}^N S_n = 0$ implies

$$P = \frac{1}{N} \sum_{n=1}^N \mathbb{E}^n[\nu] = \frac{\tau_v^{1/2}}{\tau} \left(\frac{\tau_H^{1/2} + (N-1) \tau_L^{1/2}}{N} \sum_{n=1}^N i_n \right). \quad (\text{B-2})$$

Substituting (B-2) into (B-1) yields

$$S_n = \frac{1}{A} \left(1 - \frac{1}{N} \right) \tau_v^{1/2} (\tau_H^{1/2} - \tau_L^{1/2}) (i_n - i_{-n}).$$

Thus, each trader's optimal inventory is proportional to the difference between his signal i_n and the average of the other $N-1$ signals. Equation (B-2) implies that the equilibrium price is a weighted average of traders' valuations about the fundamental value of the asset with weights summing to one.

Define the constant C_J as the ratio of the average of the square roots to the square root of the average of precisions:

$$C_J := \frac{\frac{1}{N} \tau_H^{1/2} + \frac{N-1}{N} \tau_L^{1/2}}{\left(\frac{1}{N} \tau_H + \frac{N-1}{N} \tau_L \right)^{1/2}}.$$

We can rewrite the price as

$$P = \frac{1}{N} \sum_{n=1}^N \mathbb{E}^n[\nu] = \frac{\tau_v^{1/2}}{\tau} \left(C_J \left(\frac{1}{N} \tau_H + \frac{N-1}{N} \tau_L \right)^{1/2} \sum_{n=1}^N i_n \right).$$

When traders are relatively overconfident ($\tau_H > \tau_L$), Jensen's inequality implies $C_J < 1$. Therefore, the "static dampening effect" of $C_J < 1$ shows up in our one-period model.

Under the beliefs of trader n , the auto-covariance of returns can be shown to be

$$\text{Cov}^n[\nu - P, P] = (1 - C_J^2) \frac{\tau_v}{\tau^2} (\tau_H + (N-1) \tau_L)^2 C_J^2.$$

Returns exhibit momentum when $0 < C_J < 1$, even though price is the average of traders' expectations of fundamental value. This dampening does not occur when traders have a common prior. Our dynamic model shows that the ability to engage in dynamic speculative trading based on disagreement about differences in future valuations adds additional dynamic dampening, which increases when there are more trading opportunities and may be quantitatively much greater than this static dampening based on one round of trading.

B.2 Dampening Effect: The Present Value of Expected Cumulative Dividends and Cash Flow

In this section, we discuss expectations of each trader about how his own valuation, the average valuation of other traders, and the market price evolve over time. The trader's own valuation is a martingale with respect to the trader's own filtration. Each trader believes the average valuation of other traders and market prices follow a more complicated dynamics.

In the dynamic process of sufficient statistics $dH_n(t)$ and $dH_{-n}(t)$ (equation (A-14)), the coefficient $\tau_H^{1/2}$ in the second term on the right hand side of $dH_n(t)$ is different from the coefficient $\tau_L^{1/2}$ in the second term on the right hand side of $dH_{-n}(t)$. This difference is the key driving force behind the price-dampening effect resulting from the Keynesian beauty contest. It captures the fact that—in addition to disagreeing about the value of the asset in the present—traders also disagree about the dynamics of their future valuations.

From equations (A-8) and (A-14), we can derive the stochastic process for estimates $G_n(t)$ and $G_{-n}(t) := \frac{1}{N-1} \sum_{m=1, m \neq n}^N G_m(t)$:

$$\begin{aligned} dG_n(t) &= -\alpha_G G_n(t) dt + \sigma_G \Omega^{1/2} \left(\tau_0^{1/2} dB_0^n(t) + \tau_H^{1/2} dB_n^n(t) + \tau_L^{1/2} \sum_{m=1, m \neq n}^N dB_m^n(t) \right), \\ dG_{-n}(t) &= - \left(\alpha_G + \left(\tau_H^{1/2} - \tau_L^{1/2} \right)^2 \right) G_{-n}(t) dt + \left(\tau_0 + \tau_L^{1/2} \left(2\tau_H^{1/2} + (N-2)\tau_L^{1/2} \right) \right) (G_n(t) - G_{-n}(t)) dt \\ &\quad + \sigma_G \Omega^{1/2} \left(\tau_0^{1/2} dB_0^n(t) + \tau_L^{1/2} dB_n^n(t) + \frac{\tau_H^{1/2} + (N-2)\tau_L^{1/2}}{N-1} \sum_{m=1, m \neq n}^N dB_m^n(t) \right). \end{aligned} \tag{B-3}$$

Trader n believes that his own estimate of the expected growth rate $G_n(t)$ will mean-revert to zero at a rate α_G . Trader n believes other traders' estimates of the expected growth rates $G_{-n}(t)$ will follow a more complicated path: (1) they will mean-revert to zero at a rate $\alpha_G + \left(\tau_H^{1/2} - \tau_L^{1/2} \right)^2$, faster than α_G , and (2) they will drift towards trader n 's estimate $G_n(t)$.

From equations (21), (22), and (47), at time 0, trader n 's estimates the expected value of

dividends, the long-term growth rate, his own estimate of the growth rate, and other traders' estimate of the growth rate at time $T = fh$ as follows:

$$\begin{aligned} E_0^n[D_f] &= \frac{1}{\alpha_D - \alpha_G} (e^{-\alpha_G T} - e^{-\alpha_D T}) G_{n,0} + e^{-\alpha_D T} D_0, & E_0^n[G_{n,f}] &= e^{-\alpha_G T} G_{n,0}, \\ E_0^n[G_{-n,f}] &= \frac{1}{\tau} \left(\tau_0 + \tau_L^{1/2} \left(2\tau_H^{1/2} + (N-2)\tau_L^{1/2} \right) \right) (e^{-\alpha_G T} - e^{-(\alpha_G + \tau)T}) G_{n,0} + e^{-(\alpha_G + \tau)T} G_{-n,0}. \end{aligned} \quad (B-4)$$

Suppose that trader n buys one share of the risky asset at $t = 0$, holds it until $t = T = fh$, collects all dividends, and then liquidates his position. First, assume that trader n liquidates his position at date $T = fh$ at a value equal to his own current estimate of fundamental value at T , using trader n 's estimate at $t = 0$ of what his own valuation at $t = T$, yielding

$$PV_{n,f} := E_0^n \left[\int_0^T e^{-ru} D(u) du + e^{-rT} \left(\frac{D_f}{r + \alpha_D} + \frac{G_{n,f}}{(r + \alpha_D)(r + \alpha_G)} \right) \right]. \quad (B-5)$$

Substituting (B-4) into (B-5), equation (B-5) is equal to the current trader n 's estimate of the fundamentals $F_{n,0}$,

$$PV_{n,f} = F_{n,0} = \frac{D_0}{r + \alpha_D} + \frac{G_{n,0}}{(r + \alpha_D)(r + \alpha_G)}. \quad (B-6)$$

Second, the present value of expected cumulative dividends and cash flow from liquidating one share of the stock at $t = T = fh$, using trader n 's estimate of what the average of others' valuations will be at that time, is

$$PV_{-n,f} := E_0^n \left[\int_0^T e^{-ru} D(u) du + e^{-rT} \left(\frac{D_f}{r + \alpha_D} + \frac{G_{-n,f}}{(r + \alpha_D)(r + \alpha_G)} \right) \right]. \quad (B-7)$$

This equation differs from equation (B-5) only by the last term, $G_{-n,f}$ versus $G_{n,f}$. Assuming all traders have the same valuations at the present, $G_{m,0} = G_{n,0} = \bar{G}_0$, and substituting (B-4) into (B-7), equation (B-7) is equal to

$$PV_{-n,f} = F_{n,0} + \frac{(\tau_H^{1/2} - \tau_L^{1/2})^2}{\tau(r + \alpha_G)(r + \alpha_D)} (e^{-(r + \alpha_G + \tau)T} - e^{-(r + \alpha_G)T}) G_{n,0}. \quad (B-8)$$

Since $e^{-(r + \alpha_G + \tau)T} - e^{-(r + \alpha_G)T} < 0$, the buy-and-hold valuation $PV_{-n,f}$ tends to be dampened relative to $PV_{n,f} = F_{n,0}$.

Third, the present value of expected cumulative dividends and cash flow from liquidating

one share of the stock at $t = T = fh$ at the equilibrium price P_f is

$$PV_{p,f} := E_0^n \left[\int_0^T e^{-ru} D(u) du + e^{-rT} \left(\frac{D_f}{r + \alpha_D} + \frac{C_G \bar{G}_f}{(r + \alpha_D)(r + \alpha_G)} \right) \right]. \quad (\text{B-9})$$

This equation differs from equations (B-5) and (B-7) only by the last term, containing $C_G \bar{G}_f$ instead of $G_{n,f}$ or $G_{-n,f}$. Substituting (B-4) into (B-9), it can be shown that (B-9) is equivalent to

$$PV_{p,f} = F_{n,0} + \frac{(C_G - 1)N e^{-(r + \alpha_G)T} + C_G (\tau_H^{1/2} - \tau_L^{1/2})^2 \tau^{-1} (N - 1) (e^{-(r + \alpha_G + \tau)T} - e^{-(r + \alpha_G)T})}{N (r + \alpha_G) (r + \alpha_D)} G_{n,0}. \quad (\text{B-10})$$

Since $C_G < 1$ and $e^{-(r + \alpha_G + \tau)T} - e^{-(r + \alpha_G)T} < 0$, the buy-and-hold valuation $PV_{p,f}$ tends to be dampened relative to $PV_{n,f} = F_{n,0}$ and $PV_{-n,f}$ for any f . At $t = 0$, the buy-and-hold valuation $PV_{p,0}$ is equal to the current market price P_0 . Equation (B-10) shows that the price is dampened relative to expectations about fundamentals.

B.3 Robustness Check for Model Calibration

In this subsection, we discuss how our calibration results are affected by parameter values.

There are eleven parameters in the model to be calibrated: A , r , α_D , σ_D , N , τ_H , τ_L , α_G , σ_G , $\hat{\alpha}_G$, and $\hat{\tau}$. The magnitude and horizon of return momentum are determined by the level of disagreement τ_H/τ_L , the decay rate of signals (α_G and τ), and the size of σ_G . The magnitude of long run return reversal is mainly determined by the size of σ_G and the difference between empirically correct beliefs and traders' beliefs about the mean-reversion rate $\hat{\alpha}_G - \alpha_G$ and the precision of the signals $\hat{\tau} - \tau$.

Figure B-1 shows how robust our results are to changes in parameters. Panel (a) shows that coefficient α_D has no impact on the regression coefficients. Panels (b), (c), and (d) show that when the mean reversion rate α_G is smaller (more persistent) or the volatility ratio σ_G/σ_D is larger, return momentum is more significant. Since overconfidence dampens prices, there is more momentum when the growth has a larger variance $\sigma_G^2/(2\alpha_G)$, and thus traders have more to disagree about. Panels (e) and (f) show that when the empirically correct decay rate $\hat{\alpha}_G + \hat{\tau}$ is larger (less persistent) than what traders believe ($\alpha_G + \tau$), the magnitude of return reversal increases, as expected.

To check how well alternative calibrated parameters fit the data, in Figure B-2, we set parameter values for τ_H , N , σ_D , and α_G at different levels and then calibrate parameter values of

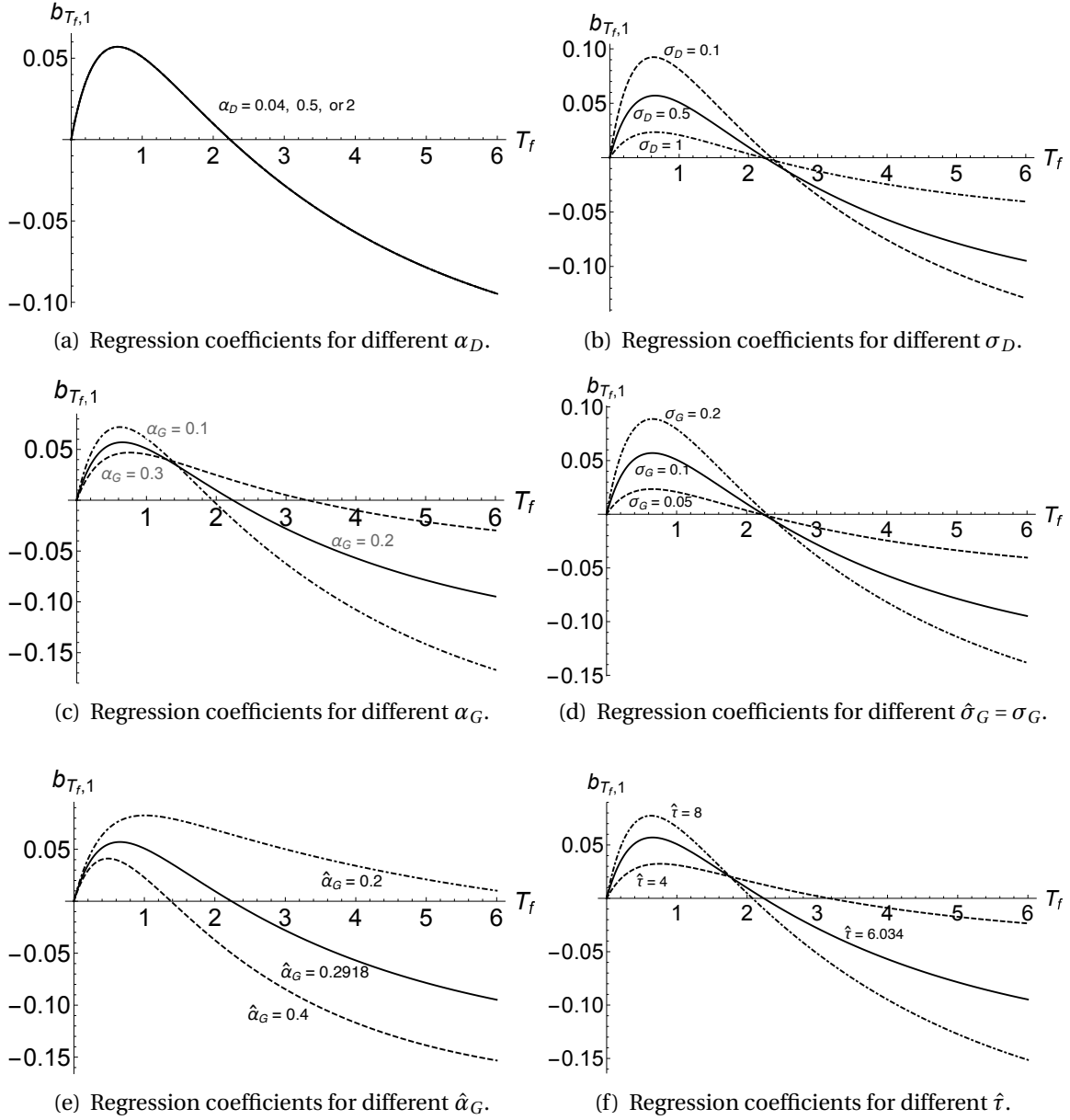


Figure B-1: Regression coefficients against T_f for different α_D ($\alpha_D = 0.04, 0.5$, and 2) and σ_D ($\sigma_D = 0.1, 0.5$, and 1), α_G ($\alpha_G = 0.1, 0.2$, and 0.3), σ_G ($\sigma_G = 0.05, 0.1$, and 0.2), $\hat{\alpha}_G$ ($\hat{\alpha}_G = 0.2, 0.2918$, and 0.4), and $\hat{\tau}$ ($\hat{\tau} = 4, 6.034$, and 8). We assume fixed parameter values $r = 0.01$, $A = 1$, $\tau_H = 0.3$, $\tau_L = 0.02$, $N = 100$, $\alpha_D = 0.04$, $\sigma_D = 0.5$, $\alpha_G = 0.2$, $\sigma_G = \hat{\sigma}_G = 0.1$ and the calibrate parameter values $\hat{\alpha}_G = 0.292$, and $\hat{\tau} = 6.034$ as default parameters.

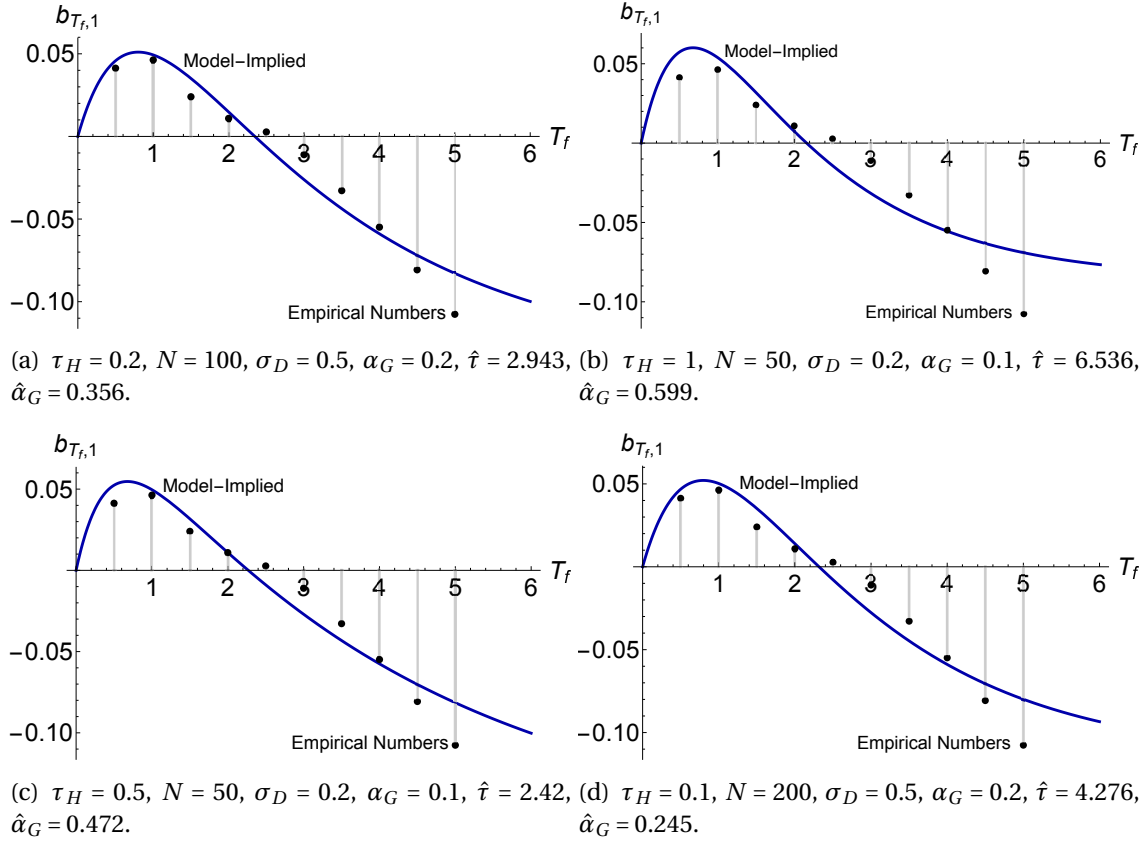


Figure B-2: Model implied (solid curve) and empirically estimated (dots) regression coefficients for different holding periods T_f . Other parameters are assumed as $r = 0.01$, $A = 1$, $\tau_L = 0.01$, $\alpha_D = 0.04$, and $\sigma_G = \hat{\sigma}_G = 0.1$.

$\hat{\alpha}_G$ and $\hat{\tau}$.

In Figure B-3, traders are assumed to have correct beliefs about mean-reversion rate of growth ($\hat{\alpha}_G = \alpha_G = 0.3$). We set parameter values for σ_D and σ_G at different levels and then calibrate parameter value of $\hat{\tau}$. The solid curves depict the theoretical regression coefficients $b_{T_f,1}$ for the calibrated parameter values. The dots correspond to empirical slope estimates. All calibrated models closely track the empirical data. The calibrated parameter values $\hat{\alpha}_G + \hat{\tau} > \alpha_G + \tau$, similar to the calibration presented in Section 4.

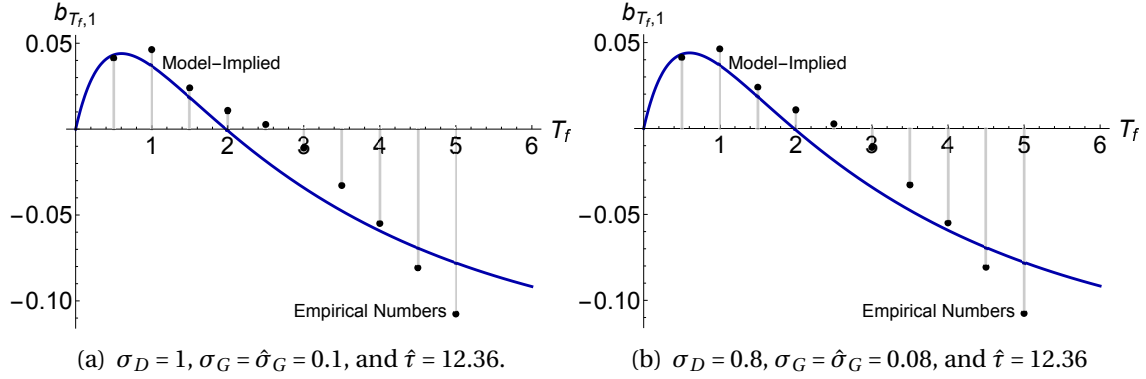


Figure B-3: Model implied (solid curve) and empirically estimated (dots) regression coefficients for different holding periods T_f . Other parameters are assumed as $r = 0.01$, $A = 1$, $\tau_H = 0.3$, $\tau_L = 0.02$, $N = 100$, $\alpha_D = 0.04$, and $\alpha_G = \hat{\alpha}_G = 0.3$.

B.4 Empirical Tests of Model Predictions

Our model predicts that time-series momentum tends to be more pronounced in securities with higher trading volume. Trading volume is a function of the trading interval h . Since inventories in the continuous model follow a diffusion, trading volume becomes infinite as $h \rightarrow 0$. When the trading interval is h , trading volume—measured as shares per unit of time—is

$$\text{Volume} := \mathbb{E} \left[\frac{N |S_n(t+h) - S_n(t)|}{h^{1/2}} \right] \frac{1}{h^{1/2}}.$$

Even though volume varies with h , the expectation $\mathbb{E} [N |S_n(t+h) - S_n(t)| h^{-1/2}]$ is a proxy for trading volume for small h because this expectation has a well-defined finite limit as $h \rightarrow 0$.

It can be shown numerically that trading volume, traders' expected inventory, and the theoretical regression coefficient $b_{6,6}$, from regressing the six-month return $R_{k,k+6}$ on the lagged six-month return $R_{k-6,k}$, increase monotonically in disagreement τ_H/τ_L . Indeed, Figure B-4 illustrates that time-series momentum tends to be stronger in markets with higher trading volume. This prediction is consistent with the theoretical finding that both factors C_G and C_J decrease with disagreement, implying more significant price dampening (as illustrated in Figure 6 of the paper).

We next test these predictions empirically. Following Campbell, Grossman, and Wang (1993) and Lee and Swaminathan (2000), we use stock turnover as an empirical proxy for trading vol-

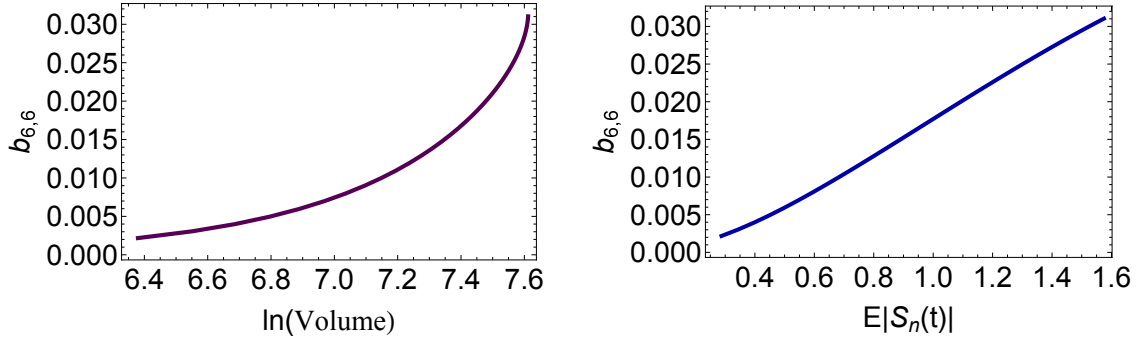


Figure B-4: Regression coefficient $b_{6,6}$ against $\ln(\text{Volume})$ and $E|S_n(t)|$ while changing τ_H/τ_L with $h = 0.01$. The parameter values are $r = 0.01$, $A = 1$, $\alpha_D = 0.04$, $\sigma_D = 0.5$, $N = 100$, $\alpha_G = 0.2$, $\sigma_G = 0.1$, $\hat{\alpha}_G = 0.2$, and vary τ_H and τ_L while holding total precision τ fixed.

ume.^{B.1} For each month from January 1965 to December 2005, stocks are sorted into five groups based on monthly turnover rate averaged over the portfolio formation period, where monthly turnover rate is the ratio of the number of shares traded each month to the number of shares outstanding at the end of the month. Let $\text{Dummy}_{k,i}$ denote a dummy variable, which is equal to one for stocks in the highest quintile and zero for stocks in the lowest quintile in month k , based on turnover in the previous six months. Stocks in the other groups are not considered.^{B.2} We then estimate the following two models using monthly Fama–MacBeth cross-sectional regressions of six-month returns on previous six-month returns:

$$\text{(Model C)} \quad R_{k,k+6,i} = a_{6,6} + b_{6,6}R_{k-6,k,i} + \varepsilon_{k,k+6,i},$$

$$\begin{aligned} \text{(Model D)} \quad R_{k,k+6,i} = & a_{6,6} + b_{6,6}R_{k-6,k,i} + c_{6,6}\text{Dummy}_{k,i} \\ & + d_{6,6}\text{Dummy}_{k,i} \times R_{k-6,k,i} + \varepsilon_{k,k+6,i}. \end{aligned}$$

Table B-1 reports the estimated slope. For Model C, the estimated slope is positive and significant, thus confirming the presence of short-run time-series momentum. For Model D, the estimated interaction term $R_{k-6,k,i} \times \text{Dummy}_{k,i}$ is positive and statistically significant at the 0.1% level, while the coefficient on $R_{k-6,k,i}$ is no longer significant. This suggests that time-series momentum is indeed more pronounced for stocks with higher trading volume (turnover),

^{B.1}As noted in Lee and Swaminathan (2000), raw trading volume is unscaled and is likely to be highly correlated with firm size. In our model of one stock, the concept of trading volume corresponds to turnover.

^{B.2}When we change the dummy variable to be equal to one for the top two quintiles and zero for the bottom two quintiles, the results are similar.

Table B-1: Regression Tests of Return Momentum for the Stocks of the Lowest Quintile and Highest Quintile Based on Trading Volume (Turnover). This table reports time-series average of slope coefficients estimated from monthly Fama–MacBeth cross-sectional regressions of models C and D from January 1965 to December 2005 for the stocks of the lowest quintile and highest quintile based on turnover. The dummy variable ($\text{Dummy}_{k,i}$) is 1 for stocks in the highest turnover quintile and 0 for stocks in the lowest turnover quintile. The coefficients are the time-series (492 months) means with t -statistics in parentheses. The standard errors of the time-series means are computed using the Hansen and Hodrick (1980) correction.

Variable	Time-Series Average Slope Coefficients			
	Model C		Model D	
	Coeff	t -stat	Coeff	t -stat
$R_{k-6,k,i}$	0.0426	(3.3724)	0.0127	(0.6454)
$\text{Dummy}_{k,i}$	.	.	-0.0297	(-2.5476)
$R_{k-6,k,i} \times \text{Dummy}_{k,i}$	.	.	0.0450	(2.3581)

as predicted.

B.5 Formulas for Autocorrelation of the Holding-Period Return

Using the definitions of $H(t)$ and $\hat{H}(t)$ in equations (37) and (50), we can write a continuous 2-vector stochastic process $y_H(t) = [H(t), \hat{H}(t)]'$ satisfying the linear stochastic differential equation

$$dy_H(t) = K_H y_H(t) dt + C_H dZ_H(t),$$

where K_H is a 2×2 matrix and C_H is a 2×2 matrix given by

$$K_H = \begin{bmatrix} -\alpha_G - \tau & \tau_0 + N \hat{\tau}_I^{1/2} \tau_I^{1/2} \\ 0 & -\alpha_S \end{bmatrix}, \quad C_H = \begin{bmatrix} \tau_0^{1/2} & \sqrt{N} \tau_I^{1/2} \\ \tau_0^{1/2} & \sqrt{N} \hat{\tau}_I^{1/2} \end{bmatrix}. \quad (\text{B-10})$$

We first derive the steady-state unconditional variance-covariance matrix of $H(t)$ and $\hat{H}(t)$. Define the steady-state unconditional variance-covariance matrix of $H(t)$ and $\hat{H}(t)$ as $Q = [[q_{11}, q_{12}], [q_{12}, q_{22}]]$. In the steady state, we have

$$K_H Q + Q K_H' + C_H C_H' = 0,$$

where K_H is a 2×2 matrix and C_H is a 2×2 matrix defined in equation (B-10). It can be shown that

$$q_{11} = \frac{\tau_0 + N \tau_I}{2(\alpha_G + \tau)} + \frac{(\tau_0 + N \hat{\tau}_I^{1/2} \tau_I^{1/2})^2}{2\alpha_S (\alpha_G + \tau)}, \quad q_{12} = \frac{\tau_0 + N \hat{\tau}_I^{1/2} \tau_I^{1/2}}{2\alpha_G}, \quad q_{22} = \frac{\tau}{2\alpha_G}.$$

We now calculate $\text{Cov}[R_{k-l,k}, R_{k,k+f}]$, where $T_l = lh$ and $T_f = fh$. Since the unconditional means of $R_{k-l,k}$ and $R_{k,k+f}$ are zero, equation (A-42) yields

$$\begin{aligned} \text{Cov}[R_{k-l,k}, R_{k,k+f}] &= \hat{\mathbb{E}}[R_{k-l,k} \times R_{k,k+f}] \\ &= \hat{\mathbb{E}}[(\zeta_2(T_l) \hat{H}_{k-l} - \zeta_1(T_l) H_{k-l} + \bar{B}_{k-l,k}) (\zeta_2(T_f) \hat{H}_k - \zeta_1(T_f) H_k)], \end{aligned}$$

where $\zeta_1(\cdot)$ and $\zeta_2(\cdot)$ are as defined in equation (A-44). Define

$$\begin{aligned} \eta &:= \frac{\tau_0 + N \hat{\tau}_I^{1/2} \tau_I^{1/2}}{\tau}, \quad c_1 := -\frac{\tau_0^{1/2}(a\eta - b)}{r + \alpha_G} + \frac{\sigma_G \Omega^{1/2} C_G \tau_0^{1/2} + \sigma_D(r + \alpha_G)}{(r + \alpha_D)(r + \alpha_G)} + \frac{a(\eta - 1)\tau_0^{1/2}}{r + \alpha_G + \tau}, \\ c_2 &:= N \left(\frac{a(\hat{\tau}_I^{1/2} \eta - \tau_I^{1/2})}{r + \alpha_G + \tau} - \frac{\hat{\tau}_I^{1/2}(a\eta - b)}{r + \alpha_G} + \frac{\sigma_G \Omega^{1/2} C_G \tau_I^{1/2}}{(r + \alpha_D)(r + \alpha_G)} \right), \quad m_1 := -\frac{a(\eta - 1)^2 \tau_0}{r + \alpha_G + \tau} \zeta_1(T_f), \\ m_2 &:= \frac{\tau_0(a\eta - b)}{r + \alpha_G} (\zeta_2(T_f) - \eta \zeta_1(T_f)), \quad m_3 := (\eta - 1) \tau_0 \left(\frac{\zeta_1(T_f)(a\eta - b)}{r + \alpha_G} - \frac{a(\zeta_2(T_f) - \eta \zeta_1(T_f))}{r + \alpha_G + \tau} \right), \\ m_4 &:= \zeta_1(T_f)(\eta - 1) \tau_0^{1/2} c_1, \quad m_5 := \frac{\zeta_2(T_f) - \eta \zeta_1(T_f)}{\zeta_1(T_f)(\eta - 1)} m_4, \quad n_1 := -\zeta_1(T_f) N \frac{a(\hat{\tau}_I^{1/2} \eta - \tau_I^{1/2})^2}{r + \alpha_G + \tau}, \\ n_2 &:= (\zeta_2(T_f) - \zeta_1(T_f) \eta) \frac{N \hat{\tau}_I(a\eta - b)}{r + \alpha_G}, \quad n_3 := N \hat{\tau}_I^{1/2} (\hat{\tau}_I^{1/2} \eta - \tau_I^{1/2}) \left(\frac{a(\zeta_1(T_f) \eta - \zeta_2(T_f))}{r + \alpha_G + \tau} + \zeta_1(T_f) \frac{a\eta - b}{r + \alpha_G} \right), \\ n_4 &:= \zeta_1(T_f) (\hat{\tau}_I^{1/2} \eta - \tau_I^{1/2}) c_2, \quad n_5 := \frac{(\eta \zeta_1(T_f) - \zeta_2(T_f)) \hat{\tau}_I^{1/2}}{\zeta_1(T_f) (\tau_I^{1/2} - \hat{\tau}_I^{1/2} \eta)} n_4. \end{aligned} \tag{B-11}$$

Then direct calculations show that $\text{Cov}[R_{k-l,k}, R_{k,k+f}]$ is a function of $T_l = lh$ and $T_f = fh$ given by

$$\begin{aligned} \text{Cov}[R_{k-l,k}, R_{k,k+f}] &= \zeta_1(T_f) (\zeta_1(T_l) (q_{11} - q_{12} \eta) - \zeta_2(T_l) (q_{12} - q_{22} \eta)) e^{-(\alpha_G + \tau) T_l} \\ &\quad + (\zeta_2(T_l) q_{22} - \zeta_1(T_l) q_{12}) (\zeta_2(T_f) - \zeta_1(T_f) \eta) e^{-\alpha_G T_l} + \frac{m_1 + n_1}{2(\alpha_G + \tau)} (1 - e^{-2(\alpha_G + \tau) T_l}) \\ &\quad + \frac{m_2 + n_2}{2\alpha_G} (1 - e^{-2\alpha_G T_l}) + \frac{m_3 + n_3}{2\alpha_G + \tau} (1 - e^{-(2\alpha_G + \tau) T_l}) + \frac{m_4 + n_4}{\alpha_G + \tau - r} (1 - e^{-(\alpha_G + \tau - r) T_l}) \\ &\quad + \frac{m_5 + n_5}{\alpha_G - r} (1 - e^{-(\alpha_G - r) T_l}). \end{aligned} \tag{B-12}$$

To calculate $\text{Corr}[R_{k-l,k}, R_{k,k+f}]$, we now calculate $\text{Var}[R_{k-l,k}]$ and $\text{Var}[R_{k,k+f}]$. Define

$$\begin{aligned} d_1 &:= \frac{a^2 \left((\eta-1)^2 \tau_0 + N(\hat{\tau}_I^{1/2} \eta - \tau_I^{1/2})^2 \right)}{(r + \alpha_G + \tau)^2}, & d_2 &:= \frac{(a\eta - b)^2 (\tau_0 + N\hat{\tau}_I)}{(r + \alpha_G)^2}, \\ d_3 &:= \frac{-2a(a\eta - b) \left(\tau_0(\eta-1) + N\hat{\tau}_I^{1/2} (\hat{\tau}_I^{1/2} \eta - \tau_I^{1/2}) \right)}{(r + \alpha_G + \tau)(r + \alpha_G)}, \\ d_4 &:= \frac{-2a \left(c_1 \tau_0^{1/2} + c_2 \tau_I^{1/2} \right) (\eta-1)}{r + \alpha_G + \tau}, & d_5 &:= \frac{2(a\eta - b)(c_1 \tau_0^{1/2} + c_2 \hat{\tau}_I^{1/2})}{r + \alpha_G}, & d_6 &:= c_1^2 + \frac{1}{N} c_2^2, \end{aligned}$$

where c_1 and c_2 are defined in Equation (B-11). Then direct calculations show

$$\begin{aligned} \text{Var}[R_{k-l,k}] &= q_{11} \zeta_1^2(T_l) + q_{22} \zeta_2^2(T_l) - 2q_{12} \zeta_1(T_l) \zeta_2(T_l) + \frac{d_1(1 - e^{-2(\alpha_G + \tau)T_l})}{2(\alpha_G + \tau)} + \frac{d_2(1 - e^{-2\alpha_G T_l})}{2\alpha_G} \\ &\quad + \frac{d_5(1 - e^{-(\alpha_G - r)T_l})}{\alpha_G - r} + \frac{d_3(1 - e^{-(2\alpha_G + \tau)T_l})}{2\alpha_G + \tau} + \frac{d_4(1 - e^{-(\alpha_G + \tau - r)T_l})}{\alpha_G + \tau - r} + \frac{d_6(e^{2rT_l} - 1)}{2r}. \end{aligned} \quad (\text{B-13})$$

Replacing $T_l = lh$ by $T_f = fh$ in Equation (B-13), we can get $\text{Var}[R_{k,k+f}]$. Then, using equations (B-12) and (B-13), we get $\text{Corr}[R_{k-l,k}, R_{k,k+f}]$. From equations (B-12) and (B-13), we obtain the regression coefficient of regressing the cumulative return $R_{k,k+f}$ on $R_{k-l,k}$ as

$$b_{f,l} = \frac{\text{Cov}[R_{k-l,k}, R_{k,k+f}]}{\text{Var}[R_{k-l,k}]}. \quad (\text{B-14})$$

B.6 A Two-Period Model with Disagreement about Drift and Diffusion Components in Signals

In this section, we consider a more general signal structure where traders might potentially disagree about both signals' coefficients on the fundamental value (drift terms) and signals' coefficients on noise (diffusion terms). We show that return momentum arises when the disagreement on drift terms dominates the disagreement on diffusion terms.

Traders can trade at times $t = 1$ and $t = 2$, and the payoff $v \sim \mathcal{N}(0, 1/\tau_v)$ is realized at $t = 3$. The total supply of the risky asset is zero. At $t = 1$ and $t = 2$, each trader n observes private signals and believes that his own private signals i_{nt} and others' private signals i_{mt} are given by

$$i_{nt} := \tau_H^{1/2} (\tau_v^{1/2} v) + e_{nt}, \quad i_{mt} := \tau_L^{1/2} (\tau_v^{1/2} v) + \tau_m^{-1/2} e_{mt},$$

with $e_{nt} \sim \mathcal{N}(0, 1)$ and $e_{mt} \sim \mathcal{N}(0, 1)$. The asset payoff v and $2N$ private signal errors $e_{1t}, \dots, e_{Nt}$, with $t = 1, 2$, are independently distributed. There is no public signal.

Let $E_t^n[\dots]$ and $\text{Var}_t^n[\dots]$ denote trader n 's expectation and variance operators conditional on all signals at period t . The projection for normally distributed random variables implies that the conditional error variance is related to signal precision τ_t at time $t = 1, 2$ by

$$\tau_t := (\text{Var}_t^n[v])^{-1} = \tau_v (1 + (\tau_H + (N-1)\tau_m\tau_L) t).$$

Trader n 's valuation at time t is

$$E_t^n[v] = \frac{\tau_v^{1/2}}{\tau_t} \left(\tau_H^{1/2} \sum_{s=1}^t i_{ns} + (N-1)\tau_m\tau_L^{1/2} \sum_{s=1}^t i_{-ns} \right),$$

which puts weight proportional to $\tau_H^{1/2}$ on own private signals i_{nt} , and to $\tau_m\tau_L^{1/2}$ on others' private signals i_{mt} , $m \neq n$.

At date t , trader n 's problem is to choose the optimal inventory S_{nt} to maximize his expected utility of terminal wealth at $t = 3$,

$$E_t^n[-e^{-A W_{n3}}], \quad \text{where} \quad W_{n3} = W_{n1} + (P_2 - P_1)S_{n1} + (v - P_2)S_{n2}. \quad (\text{B-15})$$

The prices P_1 and P_2 are set to clear markets each period.

Theorem B.1. *At $t = 2$, the price P_2 is the average of traders' expectations of the fundamental value v :*

$$P_2 = \frac{1}{N} \sum_{n=1}^N E_2^n[v] = \frac{\tau_v^{1/2}}{\tau_2} (\tau_H^{1/2} + (N-1)\tau_m\tau_L^{1/2}) \frac{1}{N} \sum_{n=1}^N (i_{n1} + i_{n2}). \quad (\text{B-16})$$

At $t = 1$, the price P_1 is a weighted average of traders' expectations of the fundamental value at $t = 1$:

$$P_1 = C_g \frac{1}{N} \sum_{n=1}^N E_1^n[v].$$

The dampening factor C_g satisfies $C_g < 1$ if and only if $\tau_m < \tau_H^{1/2}/\tau_L^{1/2}$. Define momentum as the covariance of price changes being positive:

$$\text{Cov}^n[P_2 - P_1, P_1 - P_0] > 0 \quad \text{and} \quad \text{Cov}^n[v - P_2, P_2 - P_1] > 0.$$

The equation $\text{Cov}^n[P_2 - P_1, P_1 - P_0] = 0$ has two unique positive roots, r_0 and r_1 , which satisfy $r_0 < r_1 = \tau_H^{1/2}/\tau_L^{1/2}$. Then each trader believes that returns exhibit momentum if and only if τ_m

satisfies the inequality

$$\max \left[\frac{1}{1+2(\tau_H^{1/2}-\tau_L^{1/2})(\tau_H^{1/2}+(N-1)\tau_L^{1/2})}, r_0 \right] < \tau_m < r_1 = \frac{\tau_H^{1/2}}{\tau_L^{1/2}}. \quad (\text{B-17})$$

Proof: The first-order condition for S_{n2} implies

$$S_{n2} = \frac{E_2^n[\nu] - P_2}{A \text{Var}_2^n[\nu]}.$$

The market-clearing condition $\sum_{n=1}^N S_{n2} = 0$ yields the equilibrium price P_2 in equation (B-16). The price P_2 is a weighted average of traders' valuations of the fundamental value with weights summing to one. At $t = 1$, trader n 's problem is to choose optimal inventory S_{n1} to maximize

$$E_1^n \left[-\exp \left(-A \left(W_{n1} + (P_2 - P_1) S_{n1} + \frac{(E_2^n[\nu] - P_2)^2}{2A \text{Var}_2^n[\nu]} \right) \right) \right]. \quad (\text{B-18})$$

The market-clearing condition $\sum_{n=1}^N S_{n1} = 0$ yields

$$P_1 = C_g \frac{1}{N} \sum_{n=1}^N E_1^n[\nu],$$

where the coefficient C_g has closed-form solution

$$C_g = \frac{1}{N\tau_2} \left(N\tau_1 + \tau_v(\tau_H^{1/2} + (N-1)\tau_L^{1/2}\tau_m)(\tau_H^{1/2} + (N-1)\tau_L^{1/2}) \right) - \frac{(N-1)^2\tau_v^2(\tau_H^{1/2}-\tau_L^{1/2}\tau_m)^2(\tau_H^{1/2}-\tau_L^{1/2})(\tau_H^{1/2}+(N-1)\tau_L^{1/2}\tau_m)((\tau_H^{1/2}+(N-1)\tau_L^{1/2})(\tau_H^{1/2}-\tau_L^{1/2})\tau_v/\tau_1+1-1/\tau_m)}{N\tau_2((N-1)^2\tau_v(\tau_H^{1/2}-\tau_L^{1/2}\tau_m)^2((\tau_H^{1/2}-\tau_L^{1/2})^2\tau_v/\tau_1+1+1/((N-1)\tau_m))+N^2\tau_2)}. \quad (\text{B-19})$$

Therefore, at $t = 1$, the price P_1 is a weighted average of traders' valuations of the fundamental value with weights summing to C_g .

Next compute return the auto-covariance using traders' beliefs. We obtain

$$\text{Cov}^n[\nu - P_2, P_2 - P_1] = (\tau_H^{1/2} + (N-1)\tau_L^{1/2}\tau_m)(2 - C_g + 2(\tau_H + (N-1)\tau_L\tau_m)(1 - C_g)) \times \frac{(N-1)(-\tau_H^{1/2} + \tau_L^{1/2}\tau_m)(1 - (1 + 2(\tau_H + (N-2)\tau_H^{1/2}\tau_L^{1/2} - (N-1)\tau_L))\tau_m)}{N^2\tau_m(1 + \tau_H + (N-1)\tau_L\tau_m)(1 + 2\tau_H + 2(N-1)\tau_L\tau_m)^2\tau_v}, \quad (\text{B-20})$$

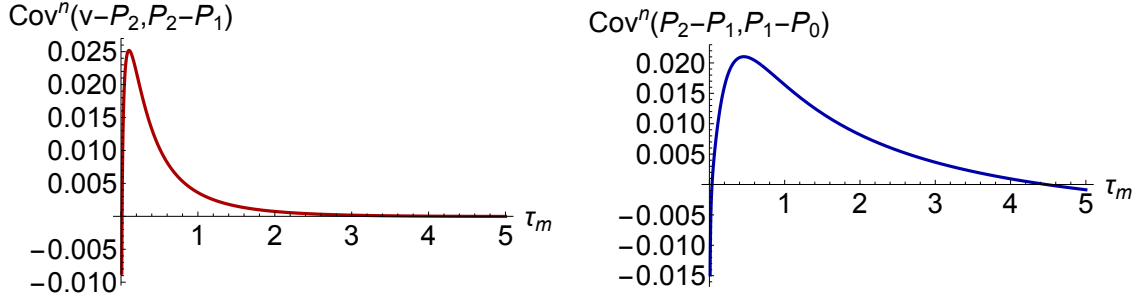


Figure B-5: Auto-covariance Under Trader n 's Belief, $\text{Cov}^n[\nu - P_2, P_2 - P_1]$ and $\text{Cov}^n[P_2 - P_1, P_1 - P_0]$.

$$\text{Cov}^n[P_2 - P_1, P_1 - P_0] = \frac{C_g(\tau_H^{1/2} + (N-1)\tau_L^{1/2}\tau_m)^2}{N^2\tau_m(1 + \tau_H + (N-1)\tau_L\tau_m)\tau_v} \times \left(\frac{N-1 + (1 + 2(\tau_H^{1/2} + (N-1)\tau_L^{1/2})^2)\tau_m}{1 + 2(\tau_H + (N-1)\tau_L\tau_m)} - \frac{C_g(N-1 + (1 + (\tau_H^{1/2} + (N-1)\tau_L^{1/2})^2)\tau_m)}{1 + \tau_H + (N-1)\tau_L\tau_m} \right). \quad (\text{B-21})$$

From equations (B-19) and (B-20), we can prove analytically that the coefficient C_g satisfies $C_g < 1$ if and only if $\tau_m < \tau_H^{1/2}/\tau_L^{1/2}$, and $\text{Cov}^n[\nu - P_2, P_2 - P_1] > 0$ if and only if the following condition holds:

$$\frac{1}{1 + 2(\tau_H^{1/2} - \tau_L^{1/2})(\tau_H^{1/2} + (N-1)\tau_L^{1/2})} < \tau_m < \frac{\tau_H^{1/2}}{\tau_L^{1/2}}.$$

There are two positive roots r_0 and $r_1 := \tau_H^{1/2}/\tau_L^{1/2}$ of the equation $\text{Cov}^n[P_2 - P_1, P_1 - P_0] = 0$. The smaller root $r_0 < 1$ has a complicated closed-form solution, which we do not provide here. Equation (B-21) implies that the auto-covariance satisfies the inequality $\text{Cov}^n[P_2 - P_1, P_1 - P_0] > 0$ if and only if $r_0 < \tau_m < r_1 = \tau_H^{1/2}/\tau_L^{1/2}$. Therefore, $\text{Cov}^n[\nu - P_2, P_2 - P_1] > 0$ and $\text{Cov}^n[P_2 - P_1, P_1 - P_0] > 0$ if and only if condition (B-17) holds. $\square$

Figure B-5 depicts return auto-covariances under traders' beliefs. It shows that return auto-covariances $\text{Cov}^n[\nu - P_2, P_2 - P_1] > 0$ and $\text{Cov}^n[P_2 - P_1, P_1 - P_0] > 0$ if and only if condition (B-17) holds.^{B.3} In this general setting with $\tau_H > \tau_L$,^{B.4} the law of iterated expectations for average expectations holds,

$$\frac{1}{N} \sum_{n=1}^N \mathbb{E}_1^n \left[\frac{1}{N} \sum_{m=1}^N \mathbb{E}_2^m[\nu] \right] = \frac{1}{N} \sum_{n=1}^N \mathbb{E}_1^n[\nu],$$

^{B.3}In this example, the condition (B-17) is $0.035 < \tau_m < 4.472$. The parameters are $N = 100$, $\tau_H = 2$, and $\tau_L = 0.1$.

^{B.4}If $\tau_H = \tau_L$, we show that the law of iterated expectations for average expectations holds for all τ_m .

if and only if $\tau_m = \tau_H^{1/2} / \tau_L^{1/2}$.

If condition (B-17) is satisfied, the law of iterated expectations for average expectations fails, dampening occurs, and there is momentum. If condition (B-17) fails with strict inequalities, the law of iterated expectations continues to fail for average expectations, price dampening does not occur, and there is mean reversion, not momentum. Price dampening and return momentum arise when the magnitude of disagreement about the variance of drift terms dominates disagreement about the variance of diffusion terms.

Under the assumptions of Section 1, condition (B-17) holds because $\tau_H > \tau_L$ and $\tau_m = 1$, implying equilibrium prices exhibit momentum.

B.7 Models without Static Dampening

In this subsection, we provide an example without static dampening but with dynamic dampening. We show that in a two-period model, returns may still exhibit momentum due to dynamic price dampening.

B.7.1 A One-Period Model

Consider a one-period model. A risky asset with random liquidation value $v \sim \mathcal{N}(0, 1/\tau_v)$ is traded for a safe numeraire asset. The total supply of the risky asset is zero. Each trader n observes a private signal. Trader n believes that his own private signal satisfies

$$i_n := \tau_H^{1/2} (\tau_v^{1/2} v) + e_n \quad \text{with} \quad e_n \sim \mathcal{N}(0, 1),$$

and other traders' private signals ($m \neq n$) satisfy

$$i_m := \frac{\tau_H^{1/2}}{N-1} (\tau_v^{1/2} v) + \tau_m^{-1/2} e_m \quad \text{with} \quad e_m \sim \mathcal{N}(0, 1), \quad (\text{B-22})$$

where

$$\tau_m := \frac{1}{1 + \frac{2(N-2)}{N-1} \tau_H}. \quad (\text{B-23})$$

For simplicity, assume that there is no public signal. Define “total precision” τ by

$$\tau := (\text{Var}^n[v])^{-1} = \tau_v \left(1 + \tau_H \left(1 + \frac{\tau_m}{N-1}\right)\right). \quad (\text{B-24})$$

This implies that trader n 's expected value of v is given by

$$E^n[v] = \frac{\tau_v^{1/2}}{\tau} \tau_H^{1/2} (i_n + \tau_m i_{-n}), \quad \text{where} \quad i_{-n} = \frac{1}{N-1} \sum_{m=1, m \neq n}^N i_m.$$

Trader n 's problem is to choose optimal inventory S_n to maximize expected utility of terminal wealth

$$E^n[-e^{-A W_n}], \quad \text{where} \quad W_n = (v - P) S_n.$$

The first-order condition for S_n implies,

$$S_n = \frac{E^n[v] - P}{A \text{Var}^n[v]}.$$

The market-clearing condition $\sum_{n=1}^N S_n = 0$ yields

$$P = \frac{1}{N} \sum_{n=1}^N E^n[v] = \frac{\tau_v^{1/2}}{\tau} \tau_H^{1/2} (1 + \tau_m) \frac{1}{N} \sum_{n=1}^N i_n. \quad (\text{B-25})$$

Under the beliefs of trader n , the auto-covariance of returns can be shown to be

$$\text{Cov}^n[v - P, P] = \frac{\tau_H}{N^2 \tau^2} (1 + \tau_m) (2N\tau - 4\tau_H \tau_v (1 + \tau_m) - \tau_v (\tau_m + N + (N-1)/\tau_m)) = 0.$$

In this example, the tendency for disagreement about drift terms to create momentum is exactly offset by the tendency for disagreement about the diffusion terms to create mean reversion. Since these two effects cancel out, each trader believes that there is neither return momentum nor return mean reversion.

B.7.2 A Two-Period Model

If we extend the model to have two trading dates, then a dynamic dampening effect arises ($C_g < 1$) and returns may exhibit momentum, even though there is no static dampening.

Assume that traders can trade in $t = 1$ and $t = 2$, and payoff $v \sim \mathcal{N}(0, 1/\tau_v)$ is realized at $t = 3$. As in the one-period model, the total supply of the risky asset is zero.

At period t ($t = 1, 2$), each trader n observes private signals. Trader n believes that his own private signals satisfy

$$i_{nt} := \tau_H^{1/2} (\tau_v^{1/2} v) + e_{nt} \quad \text{with} \quad e_{nt} \sim \mathcal{N}(0, 1),$$

and other traders' private signals ($m \neq n$) satisfy

$$i_{mt} := \frac{\tau_H^{1/2}}{N-1} (\tau_v^{1/2} v) + \tau_m^{-1/2} e_{mt} \quad \text{with} \quad e_{mt} \sim \mathcal{N}(0, 1).$$

As in the one-period model, we assume that there is no public signal. The asset payoff v and signal errors $e_{1t}, \dots, e_{Nt}$ are independently distributed. The parameter τ_m is as defined in equation (B-23).

Let $E_t^n[\dots]$ and $\text{Var}_t^n[\dots]$ denote trader n 's expectation and variance operators conditional on all signals at period t . Define "total precision" τ_t by

$$\tau_t := (\text{Var}_t^n[v])^{-1} = \tau_v \left(1 + \tau_H \left(1 + \frac{\tau_m}{N-1}\right) t\right).$$

This implies that trader n 's expected value of v at time t is given by

$$E_t^n[v] = \frac{\tau_v^{1/2}}{\tau_t} \tau_H^{1/2} \left(\sum_{t=1}^2 i_{nt} + \tau_m \sum_{t=1}^2 i_{-nt} \right), \quad \text{where} \quad i_{-nt} = \frac{1}{N-1} \sum_{m=1, m \neq n}^N i_{mt}.$$

At period t , trader n 's problem is to choose optimal inventory S_{nt} to maximize his expected utility of terminal wealth at time $t = 3$,

$$E_t^n[-e^{-A W_{n3}}], \quad \text{where} \quad W_{n3} = W_{n1} + (P_2 - P_1) S_{n1} + (v - P_2) S_{n2}.$$

The prices P_1 and P_2 are set to clear markets each period.

Theorem B.2. *At $t = 2$, the price P_2 is the average of traders' expectations of the fundamental value v :*

$$P_2 = \frac{1}{N} \sum_{n=1}^N E_2^n[v] = \frac{\tau_v^{1/2}}{\tau_2} \tau_H^{1/2} (1 + \tau_m) \frac{1}{N} \sum_{n=1}^N (i_{n1} + i_{n2}). \quad (\text{B-26})$$

At $t = 1$, the price P_1 is a weighted average of traders' expectations of the fundamental value v with weights summing to C_g , which satisfies $C_g < 1$:

$$P_1 = C_g \frac{1}{N} \sum_{n=1}^N E_1^n[v],$$

The coefficient $C_g < 1$ is defined in (B-28). Each trader believes that prices exhibit mean reversion in the first period and momentum in the second period, defined as

$$\text{Cov}^n[P_2 - P_1, P_1 - P_0] < 0, \quad \text{and} \quad \text{Cov}^n[v - P_2, P_2 - P_1] > 0.$$

Assuming traders are correct on average, under empirically correct parameters which give all private signals the same precision, we show analytically that prices exhibit momentum in both periods:

$$\text{Cov}[P_2 - P_1, P_1 - P_0] > 0, \quad \text{and} \quad \text{Cov}[\nu - P_2, P_2 - P_1] > 0.$$

Proof: The first-order condition for S_{n2} implies

$$S_{n2} = \frac{E_2^n[\nu] - P_2}{A \text{Var}_2^n[\nu]}.$$

The market-clearing condition $\sum_{n=1}^N S_{n2} = 0$ yields the equilibrium price P_2 in equation (B-26). The price P_2 is a weighted average of traders' valuations of the fundamental value with weights summing to one. At $t = 1$, trader n 's problem is to choose optimal inventory S_{n1} to maximize

$$E_1^n \left[-\exp \left(-A \left(W_{n1} + (P_2 - P_1) S_{n1} + \frac{(E_2^n[\nu] - P_2)^2}{2A \text{Var}_2^n[\nu]} \right) \right) \right]. \quad (\text{B-27})$$

Define

$$y := [i_{n2} - E_1^n[i_{n2}], i_{-n,2} - E_1^n[i_{-n,2}]]'.$$

Then the solution for the problem (B-27) can be obtained using

$$E_1^n \left[e^{-\alpha \left(\bar{A} + B' y + \frac{1}{2} y' C y \right)} \right] = \frac{1}{\sqrt{|I + \alpha C \Sigma|}} e^{-\alpha \left(\bar{A} - \frac{1}{2} \alpha B' \Sigma (I + \alpha C \Sigma)^{-1} B \right)},$$

where y is a 2×1 vector of zero-mean normally distributed random variables with covariance matrix Σ , $\bar{A}$ is a scalar, B is a 2×1 vector, C is a 2×2 symmetric matrix, I is the 2×2 identity matrix. Define α , $\bar{A}$, B , and C as

$$\begin{aligned} \bar{A} &:= W_{n1} + (E_1^n[P_2] - P_1) S_{n1} + \frac{c}{2A} \left(E_1^n \left[\sum_{t=1}^2 i_{nt} \right] - E_1^n \left[\sum_{t=1}^2 i_{-nt} \right] \right)^2, \\ \text{with} \quad c &:= \frac{\tau_v \tau_H (N-1-\tau_m)^2}{N^2 \tau_2}, \quad C := \frac{c}{A} \times \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \\ \alpha &:= A, \quad B := \psi_B S_{n1} + \phi_B, \quad \phi_B := \left(E_1^n \left[\sum_{t=1}^2 i_{nt} \right] - E_1^n \left[\sum_{t=1}^2 i_{-nt} \right] \right) \times \left[\frac{c}{A}, -\frac{c}{A} \right]', \\ \psi_B &:= \frac{\tau_v^{1/2} \tau_H^{1/2} (1+\tau_m)}{N \tau_2} \times [1, N-1]'. \end{aligned}$$

The first-order condition for S_{n1} implies

$$S_{n1} = \frac{E_1^n[P_2] - P_1 - A\phi_B'\Sigma(I + AC\Sigma)^{-1}\psi_B}{A\psi_B'\Sigma(I + AC\Sigma)^{-1}\psi_B}.$$

The market-clearing condition $\sum_{n=1}^N S_{n1} = 0$ yields

$$P_1 = \frac{1}{N} \sum_{n=1}^N E_1^n[P_2] - [c, -c]\Sigma(I + AC\Sigma)^{-1}\psi_B \frac{N-2}{N-1} \tau_H^{1/2} \tau_v^{1/2} \frac{1}{N} \sum_{n=1}^N E_1^n[v].$$

We can show that P_1 has the form $C_g \times \frac{1}{N} \sum_{n=1}^N E_1^n(v)$, and we can analytically prove

$$C_g = \frac{\tau_1}{\tau_2} + \frac{2\tau_H\tau_v(1+\tau_m)}{N\tau_2} - [c, -c]\Sigma(I + AC\Sigma)^{-1}\psi_B \frac{N-2}{N-1} \tau_H^{1/2} \tau_v^{1/2} < 1. \quad (\text{B-28})$$

Therefore, at $t = 1$, the price P_1 is a weighted average of traders' valuations of the fundamental value with weights summing to less than one.

We next compute return auto-covariance using traders' beliefs. We can analytically prove

$$\begin{aligned} \text{Cov}^n[v - P_2, P_2 - P_1] &= \text{Cov}^n\left[v - \frac{1}{N} \sum_{n=1}^N E_2^n[v], \frac{1}{N} \sum_{n=1}^N E_2^n[v] - C_g \times \frac{1}{N} \sum_{n=1}^N E_1^n(v)\right] \\ &= \frac{\tau_H(1+\tau_m)(N-1-\tau_m)(N-1)((2-C_g)(N-1)+2\tau_H(1-C_g)(N-1+\tau_m))(1-N+\tau_m(-1+4\tau_H(N-2)+N))}{\tau_m N^2(N-1+\tau_H(\tau_m+N-1))(N-1+2\tau_H(N-1+\tau_m))^2 \tau_v} > 0, \end{aligned}$$

$$\begin{aligned} \text{Cov}^n[P_2 - P_1, P_1 - P_0] &= -\frac{\tau_H C_g (1+\tau_m)^2 (N-1)^2}{\tau_m N^2(N-1+\tau_H(\tau_m+N-1))^2(N-1+2\tau_H(N-1+\tau_m))\tau_v} \times \\ &\quad (\tau_H(\tau_m^2(2C_g-1) + (N-1)(2C_g(4\tau_m+N-1) - (10\tau_m+N-1))) + (N-1+\tau_m)(C_g-1)(8\tau_H^2\tau_m+N-1)) < 0. \end{aligned}$$

Traders believe that prices exhibit mean reversion in the first period because the disagreement about diffusion terms dominates disagreement about drift terms in this specific two-period model with the parameter τ_m defined in equation (B-23). At time $t = 2$, the dynamic dampening effect dominates and the return exhibits momentum.

Assume traders are correct on average, under the empirically correct parameters that all private signals have the same precision, traders' private signals satisfy

$$\hat{i}_{nt} := \left(\frac{1}{N}\tau_H\left(1 + \frac{\tau_m}{N-1}\right)\right)^{1/2} (\tau_v^{1/2} v) + \hat{e}_{nt} \quad \text{with} \quad \hat{e}_{nt} \sim \mathcal{N}(0, 1). \quad (\text{B-29})$$

Under empirically correct (objective) parameters, prices are

$$\begin{aligned} P_1 &= C_g \frac{1}{N} \sum_{n=1}^N E_1^n[\nu] = C_g \frac{\tau_v^{1/2} \tau_H^{1/2}}{\tau_1} (1 + \tau_m) \frac{1}{N} \sum_{i=1}^n \hat{i}_{n1}, \\ P_2 &= \frac{1}{N} \sum_{n=1}^N E_2^n[\nu] = \frac{\tau_v^{1/2} \tau_H^{1/2}}{\tau_2} (1 + \tau_m) \left(\frac{1}{N} \sum_{i=1}^n \hat{i}_{n1} + \frac{1}{N} \sum_{i=1}^n \hat{i}_{n2} \right). \end{aligned} \quad (\text{B-30})$$

Substituting Equation (B-29) into Equation (B-30), we obtain the return auto-covariances

$$\begin{aligned} \text{Cov}[P_2 - P_1, P_1 - P_0] &= \frac{C_g \tau_H \tau_v (1 + \tau_m)^2}{N \tau_1} \left(\tau_H \left(1 + \frac{\tau_m}{N-1} \right) \left(\frac{2}{\tau_2} - \frac{C_g}{\tau_1} \right) + \frac{1}{\tau_2} - \frac{C_g}{\tau_1} \right) > 0. \\ \text{Cov}[\nu - P_2, P_2 - P_1] &= \tau_H^{1/2} (1 + \tau_m) \left(\frac{2}{\tau_2} - \frac{C_g}{\tau_1} \right) \left(\left(\frac{1}{N} \tau_H \left(1 + \frac{\tau_m}{N-1} \right) \right)^{1/2} - \tau_H^{1/2} (1 + \tau_m) / N \right) > 0. \end{aligned}$$

□

To summarize, in this example, there is no static dampening. Yet, due to dynamic dampening, returns nevertheless exhibit momentum under empirically correct parameters.

B.8 Return Autocorrelations Using Traders' Beliefs

In this subsection, we use traders' beliefs to study return dynamics. We show that dynamic price dampening leads to positive return autocorrelation, implying that returns exhibit momentum. In addition, return momentum is greater when traders can trade more frequently. Trader n believes that the dynamics of $\bar{G}(t) := \frac{1}{N} \sum_{n=1}^N G_n(t)$ and $G_n(t)$ are given by

$$\begin{aligned} d\bar{G}(t) &= -(\alpha_G + \tau) \bar{G}(t) dt + (\tau_0 + N \tau_I) G_n(t) dt \\ &\quad + \sigma_G \Omega^{1/2} \tau_I^{1/2} \left(dB_n^n(t) + N \hat{a} dB_0^n(t) + \sum_{m=1, m \neq n}^N dB_m^n(t) \right), \end{aligned} \quad (\text{B-31})$$

$$dG_n(t) = -\alpha_G G_n(t) dt + \sigma_G \Omega^{1/2} \left(\tau_H^{1/2} dB_n^n(t) + \tau_0^{1/2} dB_0^n(t) + \tau_L^{1/2} \sum_{m=1, m \neq n}^N dB_m^n(t) \right), \quad (\text{B-32})$$

where $\hat{a} := \tau_0^{1/2} / (\tau_H^{1/2} + (N-1) \tau_L^{1/2})$ and $\tau_I^{1/2} := \frac{1}{N} \tau_H^{1/2} + \frac{N-1}{N} \tau_L^{1/2}$. Trader n believes that the instantaneous returns process is a linear combination of $G_n(t)$ and $\bar{G}(t)$:

$$\begin{aligned} dP(t) &+ D(t) dt - r P(t) dt \\ &= \left(\frac{1}{r + \alpha_D} + \frac{C_G(\tau_0 + N \tau_I)}{(r + \alpha_D)(r + \alpha_G)} \right) G_n(t) dt - C_G \left(\frac{1}{r + \alpha_D} + \frac{\tau}{(r + \alpha_D)(r + \alpha_G)} \right) \bar{G}(t) dt + dB_r(t), \end{aligned} \quad (\text{B-33})$$

where $dB_r(t)$ is defined as follows:

$$dB_r(t) := \frac{\sigma_D}{r + \alpha_D} dB_0^n(t) + \frac{\sigma_G C_G \Omega^{1/2}}{(r + \alpha_D)(r + \alpha_G)} \left(\tau_0^{1/2} dB_0^n(t) + \tau_I^{1/2} \left(dB_n^n(t) + \sum_{m=1, m \neq n}^N dB_m^n(t) \right) \right).$$

The holding period return from $t = kh$ to $t + T = (k + f)h$ is

$$R_{k,k+f} = \int_{u=kh}^{kh+T} e^{r(kh+T-u)} (dP(u) + D(u) du - rP(u) du). \quad (B-34)$$

Define a continuous 2-vector stochastic process $y_G(t) = [G_n(t), \bar{G}(t)]'$. It can be shown that $y_G(t)$ satisfies the linear stochastic differential equation $dy_G(t) = K_G y_G(t) dt + C_Z dZ_G(t)$, where K_G is a 2×2 matrix and C_Z is a 2×3 matrix given by

$$K_G = \begin{bmatrix} -\alpha_G & 0 \\ \tau_0 + N\tau_I & -(\alpha_G + \tau) \end{bmatrix}, \quad C_Z = \sigma_G \Omega^{1/2} \begin{bmatrix} \tau_0^{1/2} & \tau_H^{1/2} & (N-1)^{1/2} \tau_L^{1/2} \\ \tau_0^{1/2} & \tau_I^{1/2} & (N-1)^{1/2} \tau_I^{1/2} \end{bmatrix}.$$

The vector $dZ_G(t) = [dB_0^n(t), dB_n^n(t), (N-1)^{-1/2} \sum_{m=1, m \neq n}^N dB_m^n(t)]'$ is a 3×1 -dimensional Brownian motion. We can then obtain recursive formulas for the stochastic vector $y_G(u) = [G_n(u), \bar{G}(u)]'$ for time $u \geq kh$ as a function of $y_G(kh) = [G_{n,k}, \bar{G}_k]'$. Plugging these recursive formulas into equation (B-34), we obtain the cumulative holding period return $R_{k,k+f}$ over period $T = fh$ between time kh and $(k + f)h$ as a linear function of $G_{n,k}$ and $\bar{G}_k$:

$$R_{k,k+f} = \zeta_2(T) G_{n,k} - \zeta_1(T) \bar{G}_k + \bar{B}_{k,k+f}. \quad (B-35)$$

The coefficients $\zeta_1(T) > 0$, $\zeta_2(T) > 0$, and $\bar{B}_{k,k+f}$ in the equation above are defined as

$$\begin{aligned} \zeta_1(T) &:= \frac{C_G}{(r + \alpha_G)(r + \alpha_D)} (e^{rT} - e^{-(\alpha_G + \tau)T}), \\ \zeta_2(T) &:= \frac{e^{-(\alpha_G + \tau)T}}{(r + \alpha_G)(r + \alpha_D)\tau} ((e^{(r + \alpha_G + \tau)T} - e^{rT})\tau + C_G(\tau_0 + N\tau_I)(e^{rT} - 1)), \end{aligned} \quad (B-36)$$

$$\bar{B}_{k,k+f} := \int_{s=kh}^{kh+T} \int_{u=s}^{kh+T} [a, -b] e^{r(kh+T-u) + K_G(u-s)} C_Z du dZ_G(s) + \int_{s=kh}^{kh+T} e^{r(kh+T-s)} dB_r(s),$$

where constants a and b are defined as follows:

$$a := \frac{1}{r + \alpha_D} + \frac{C_G(\tau_0 + N\tau_I)}{(r + \alpha_D)(r + \alpha_G)}, \quad b := C_G \left(\frac{1}{r + \alpha_D} + \frac{\tau}{(r + \alpha_D)(r + \alpha_G)} \right). \quad (B-37)$$

Figure B-6 depicts $\text{Corr}^n[R_{k-f,k}, R_{k,k+f}]$ for $T = fh$; plots are presented for markets with

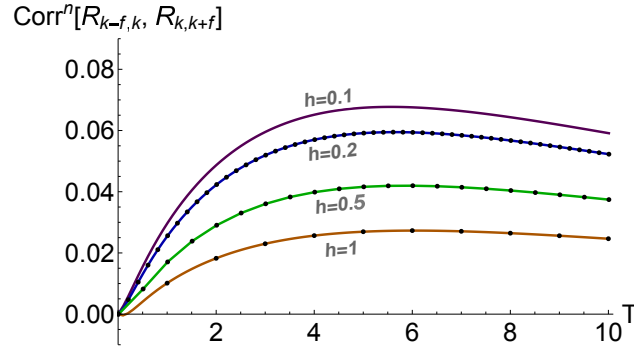


Figure B-6: Autocorrelations of Holding-Period Returns Under Trader n 's Beliefs, $\text{Corr}^n[R_{k-f,k}, R_{k,k+f}]$ for $T = fh$. The parameter values of r , A , N , α_D , σ_D , α_G , σ_G , τ_H , and τ_L are given in Table 2 in Section 4 of the paper.

different trading intervals $h = 0.1, 0.2, 0.5$, and 1 . The return autocorrelations are positive and increasing with trading frequency, implying that the time-series return momentum is more significant when h is smaller and traders trade more frequently.